16/10/12

Unit - I

System :- A System is a Combination of an arrangement of different physical Components which act together as an entire Unit to achieve Certain objective.

Every physical object is actually a system. A

A classroom is a good example of physical system. A room along with the Combination of benches, blackboard, fans, lighting System etc.

Control System :-

When a number of elements or components, that are Connected in a sequence to perform a specific function, is known as System.

In a System when the output quantity is controlled by the input quantity, then the System is called control System. Here, the output quantity is known as Controlled Variable (or) Response and input quantity is Called the Command signal (or) Excitation.

*** Classification of Control Systems :-** [U.A BAKSHI, V.U.BAKSHI]

1. Natural Control Systems :-

The biological systems, systems inside human being are of Natural type

2. Manmade Control Systems :-

The various systems, we are using in our daily day-to-day life are designed and manufactured by human beings. Systems like vehicles, switches, etc are called manmade control systems.

3. Combinational Control Systems :-

Combinational control system is one, having combination of natural and manmade together i.e Driver driving a vehicle.

→ But for engineering analysis, control systems are classified in many different ways. Some of the classifications are below.

4. Time Variant and Time Invariant Systems :-

Time varying control systems are those in which parameters of the system are varying with time.

Time invariant systems are those, the inputs and outputs are functions of time, but the parameters of the system are independent of time, which are not varying with time and are constants.

5. Linear and Non-linear Systems :-

A control system is said to be linear if it satisfies the following properties.

(a) The Principle of Superposition is applicable to the System. This means the Response of several inputs can be obtained by considering one input at a time and then algebraically adding the invidual Results.

Mathematically principle of superposition is expressed in two ways.

i, Additive property $\Rightarrow f(x+y) = f(x) + f(y)$.

ii, Homogenicus property $\Rightarrow f(\alpha x) = \alpha f(x)$.

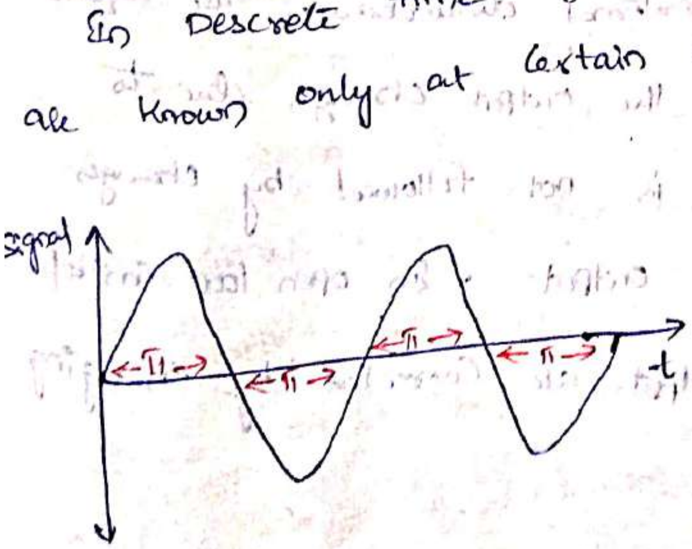
(b) The differential equation describing the system is linear having its coefficients as constants.

(c) practically the output varies linearly as the input varies.

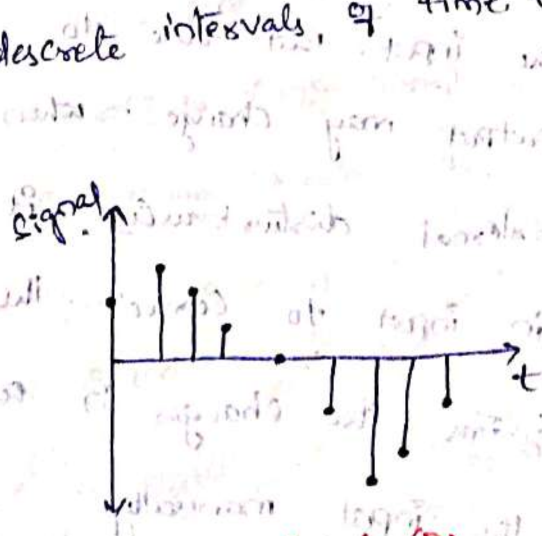
6. Continous time and Descrete time Control systems :-

In a Continous type of Control systems, all the system variables are the functions of continous time variable 't'.

In Descrete time systems one or more system variables are known only at certain descrete intervals of time 't'.



(a) Continous signal



(b) Descrete signal (or) Non-Continous signal

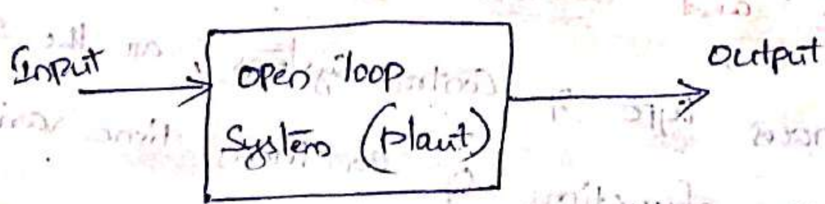
7. Open loop and closed loop Control Systems :-

A System which doesn't have a feedback element to get the desired output is known as open loop C.S.

A System having a feedback element is known as closed loop Control system.

* Open loop Control Systems :-

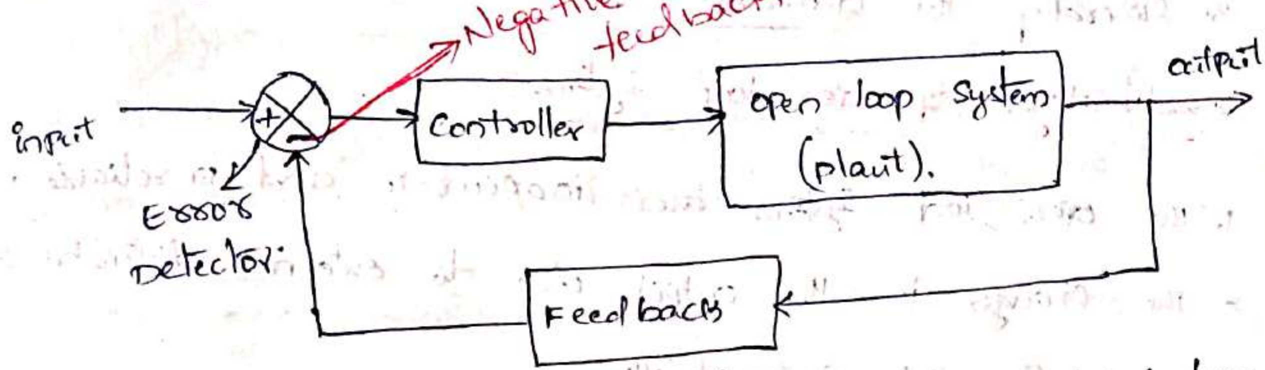
Any physical System which does not automatically Correct the variations in its output is called open loop System. (or) A Control System in which the output quantity has no effect up-on the input quantity are called open loop Control systems. This means the output is not feedback to the input for correction.



In open loop C.S the output is varied by varying the input. But due to external disturbances the System output may change. When the output changes due to external disturbances, it is not followed by changes in input to correct the output. In open loop control systems the changes in output are Corrected by changing the input manually.

* Closed loop Systems :-

Control Systems in which the output has an effect upon the input quantity in such a manner so as to maintain the desired output value are called closed loop Control Systems.



The open loop System Can be modified as closed loop System by providing a feedback. The addition of feedback element automatically corrects the changes in output due to disturbances. Hence the closed loop Systems are also Called as Automatic Control System.

The Reference (or) input signal corresponds to desired output. The feedback path samples the output and converts it to a signal of same type as that of Reference signal. The feedback signal is proportional to output signal and is fed to the Error Detector. The Error Detector compares the difference between the Reference signal and feedback signal. The Controller modifies and Amplifies the error signal to produce better action. The modified error signal is fed to the plant to Correct its output

*** Advantages of open loop Systems :-**

1. The open loop Systems are simple and economical.
2. The open loop Systems are easier to Construct.
3. Generally the open loop Systems are stable.

Disadvantages of open loop Systems :-

1. The open loop Systems are inaccurate and unreliable.
2. The changes in the output due to external disturbances are not Corrected automatically

*** Advantages of Closed loop Systems :-**

1. The Closed loop Systems are accurate and Reliable.
2. These are less affected by Noise.
3. The closed loop Systems are accurate even in the presence of Non-linearities

Disadvantages of Closed loop Systems :-

1. The closed loop Systems are Complex and costlier.
2. The feedbacks Reduces the over all gain of the system
3. The feedbacks in closed loop may lead to the oscillatory Response

NOTE :-

Among those Number of Classifications the two Major type of Control Systems are open loop and closed loop type of Control Systems.

In Case of Closed loop Control System the feed back element is employed and it is mainly used for the Automatic Correction of any changes in the output.

* Generally, the Negative feedback is employed in a Control Systems.

The Negative feed back Results in better stability and it Rejects any disturbance signals. Therefore, negative feedback is Preferred in closed loop systems.

* Also, the Characteristics of Negative feedback are,

i, Rejection of External Disturbance signals.

ii, Low sensitivity to parameter variations.

iii, Reduction in gain at the expense of better stability.

Effect of Positive feedback :-

→ The positive feedback increases the Error signal and drives the output to instability. Some times, the positive feed back signal also uses in Control systems to amplify certain internal signals (or) system parameters.

* Distinguish open loop and closed loop systems.

open loop	closed loop
i, Inaccurate & un Reliable	i, Accurate & Reliable
ii, Simple and economical	ii, Complex and Costlier.
iii, External disturbances are not Corrected automatically,	iii, External disturbances are Corrected automatically.
(iv) They are generally stable	(iv) Great efforts are needed (Care) to design a stable system.

NOTE:-

1. Since the open loop control systems does not having the capable of correcting its output automatically, since they are known as Manually Controlled Systems

2. The closed loop control systems are having the capable of correcting its output for various variations (External disturbances). So, They are known as Automatic Controlled Systems

** 3. Depending upon the number of inputs and outputs, the control systems are classified of two types. They are

1. Single input and single o/p control systems (SISO)
2. Multi input and multi output control systems (MIMO).

* Examples for open loop and closed loop control systems:-

@ Traffic control system:-

open loop system:-

Traffic control by means of traffic signals operated on a time basis constitutes an open-loop control system. The sequence of control signals are based on a time slot given for each signal. The time slots are decided based on a traffic study. The system will not measure the density of of traffic. since, the time slots does not change according to the traffic density.

Closed loop System:-

Traffic Control System can be made as a closed loop system if the time slots are decided based on the traffic density. In this type of system, the traffic density will be measured from all the sides and the Information is fed to the Computer. The timings of signals are decided by the Computer based on the traffic density. Since the closed loop system dynamically changes the timings of open loop Control system.
traffic signals will be better than

* washing Machine :-

A washing Machine without any cleanliness
open loop Control
Measuring system is an example of
washer will be
system. Here, the washing, Rinsing in the
The machine does not measure
operates on a time basis.
the cleanliness of clothes.
the output signal, that is
the Machine will
Once the set ON Time is over,
whatever may be the level of the
automatically stops,
cleanliness of clothes.
closed loop Control System, if
This can be a
can be measured and compared
the level of cleanliness
with the desired cleanliness.

* TRANSFER FUNCTION :-

The Transfer function of a system is defined as the ratio of Laplace Transform of Output to the laplace transform of input with zero initial conditions.

It is also defined as the Laplace transform of Impulse Response of a system with zero initial conditions.

⇒ The Transfer function Technique is not applicable to the Non-linear System. Also, the Transfer function of a system is independent of input and is dependent only on system parameters.

* Mathematical Model :-

The input output Relations of various physical components of a system are governed by the differential equations.

The Mathematical Model of a system constitutes a set of Differential Equations.

A Mathematical Model will be linear if the differential Equations describing (or) governing the system having the Constant Coefficients

If the Coefficients of differential equation governing the system are Constants then the system is linear time invariant If the Coefficients of differential Equation governing the system are the functions of time then the Model is known as linear time varying.

* Mechanical Translational Systems :-

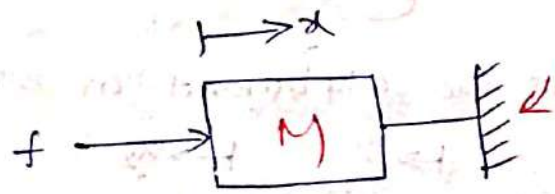
The Translational system can be obtained using three basic elements namely, MASS, SPRING, DASH-POT.

The weight of the mechanical system is represented by the element mass and it is assumed to be concentrated at the centre of the body.

The Elastic deformation of the body can be represented by a spring.

The friction existing in mechanical system can be represented by the element Dash-pot.

* Consider an element MASS which is shown below, which has Negligible friction and elasticity



let a force "f" applied on it, it offers an opposing force which is proportional to acceleration of the body

let, f = force applied

x = displacement of a body.

Now, opposing force offered by Mass $f_m \propto \frac{d^2x}{dt^2}$

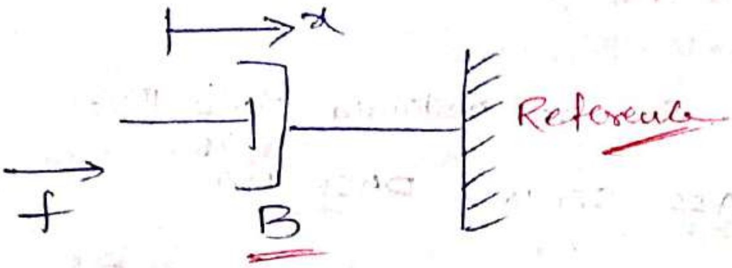
$$\Rightarrow \boxed{f_m = M\frac{d^2x}{dt^2}}$$

$\because$ A/c to Newtons law, $f = ma$

$a = \frac{dv}{dt}$ again $v = \frac{dx}{dt}$

$\therefore a = \frac{d^2x}{dt^2}$ $\Rightarrow \therefore f = M\frac{d^2x}{dt^2}$

Similarly, Consider an Element Dashpot Shown in figure, which is free from mass and elasticity.



let "f" be the applied force on element Dashpot, and it offers an opposing force which is Proportional to the rate of displacement of body i.e, velocity of the body.

$\therefore$ opposing force offered by Dashpot $= f_b \propto \frac{dx}{dt}$

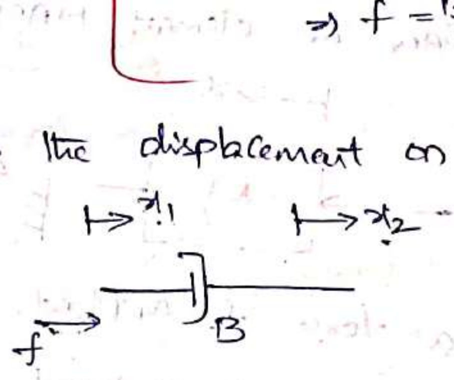
$$\Rightarrow \boxed{f_b = B\cdot\frac{dx}{dt}}$$

$\because v = \frac{dx}{dt}$

$\therefore f \propto v$

$\Rightarrow f = B\cdot\frac{dx}{dt}$

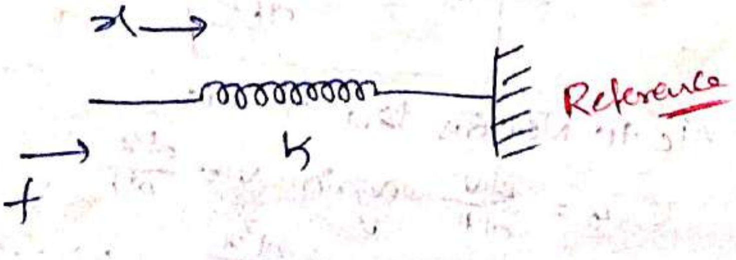
Now, Consider the dash pot has the displacement on both sides as shown below,



$$\therefore f_b = B\frac{d}{dt}(x_1 - x_2)$$

$$\therefore \boxed{f_b = B\frac{dx_1}{dt} - B\cdot\frac{dx_2}{dt}}$$

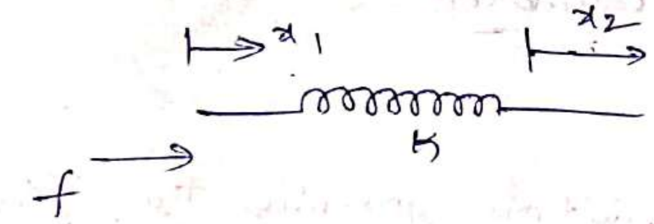
Similarly, for Elastic element shown in figure below, which is free from Mass and friction.



let a force applied on a spring, and it will offers an opposing force which is directly proportional to displacement

$$\therefore \boxed{f_K = K \cdot x}$$

⇒ when the spring has displacement on both sides as shown below, then:



Now, $f_K \propto (x_1 - x_2)$

$$\Rightarrow \boxed{f_K = K(x_1 - x_2)}$$

*Procedure for the Transfer function of Mechanical Translational System :-

1. In Mechanical translational System, the differential equations governing the system are obtained by writing force balance equations of Masses in the System.

2. The displacement at the Masses are assumed as $x_1, x_2 \ldots$ and assigning each displacement at each Mass. The first derivative of the displacement is velocity and the Second derivative of displacement is acceleration.

3. Draw the force body diagrams of the System. The force body diagram is obtained by drawing each mass seperately and then Marking all the forces acting on that Mass.

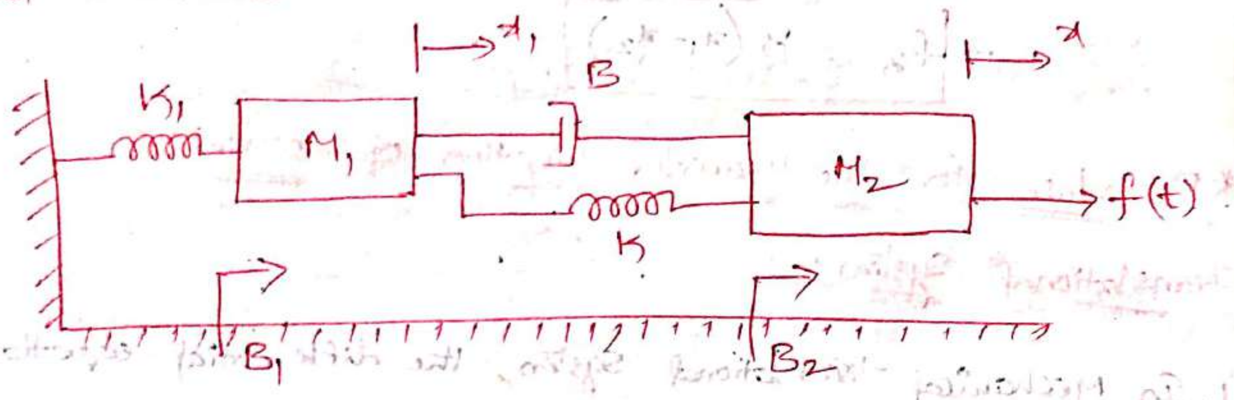
Always the opposing force acts in a direction opposite to the applied force.

4. For each free body diagram write one differential equation by equating the sum of applied forces to the sum of opposing forces.

5. Take laplace transform of differential equations to convert them to Algebraic Equations. And, finally the transfer function is to be calculated.

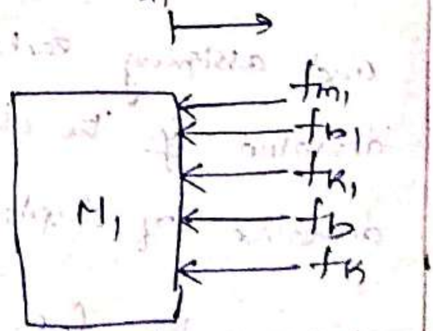
* Problems :-

1. write the differential equations governing the system and Determine the transfer function.



Sol: from the above translational system, $f(t)$ is the input and the displacement "x" be the output.

Hence, there are two Masses so, the two free body diagrams will be needed.



$\therefore$ By Newtons second law;

applied force = opposing forces

$$\Rightarrow f_{m_1} + f_{b_1} + f_b + f_{K_1} + f_K = 0$$

$$\Rightarrow f_{m_1} = M_1 \frac{d^2 x_1}{dt^2} \; ; \; f_{b_1} = B_1 \frac{dx_1}{dt} \; , \; f_b = B \frac{d}{dt}(x_1 - x) \; ;$$

$$f_{K_1} = K_1 x_1 \; ; \; f_K = K(x_1 - x)$$

$$\therefore M_1 \frac{d^2x_1}{dt^2} + B_1 \frac{dx_1}{dt} + B\frac{d}{dt}(x_1 - x) + K_1 x_1 + K(x_1 - x) = 0 \quad \text{—— (A)}$$

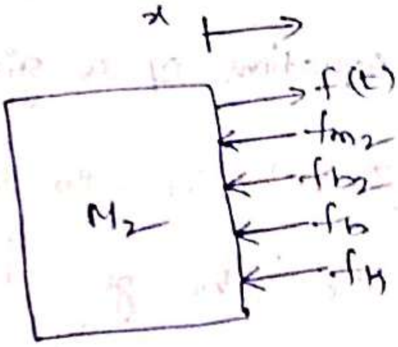
Now, taking the laplace transform we get,

$$\Rightarrow M_1 s^2 X_1(S) + B_1 s X_1(S) + Bs\left[X_1(S) - X(S)\right] + K_1 X_1(S) + K\left[X_1(S) - X(S)\right] = 0$$

$$\Rightarrow X_1(S)\left[M_1 s^2 + (B_1 + B)s + (K_1 + K)\right] = X(s)(Bs + K)$$

$$\Rightarrow \boxed{X_1(S) = \frac{Bs + K}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} X(S).} \quad \text{—— (1)}$$

for, Mass-2



$$f_{m_2} = M_2 \frac{d^2x}{dt^2} \; ; \; f_{b_2} = B_2 \frac{dx}{dt}$$

$$f_b = B\frac{d}{dt}(x - x_1) \; ; \; f_k = K(x - x_1)$$

By Newtons 2nd law,

$$f_{m_2} + f_{b_2} + f_b + f_k = f(t)$$

$$\Rightarrow M_2 \frac{d^2x}{dt^2} + B_2 \frac{dx}{dt} + B\frac{d}{dt}(x - x_1) + K(x - x_1) = f(t). \quad \text{—— (B)}$$

Applying, the laplace transform to above equation,

$$\Rightarrow M_2 s^2 X(S) + B_2 s X(S) + Bs\left[X(S) - X_1(S)\right] + K\left[X(S) - X_1(S)\right] = F(S)$$

$$\Rightarrow X(S)\left[M_2 s^2 + B_2 s + Bs + K\right] - X_1(S)\left[Bs + K\right] = F(S) \quad \text{—— (2)}$$

Now, Substituting the Equation (1) in eq (2) we get.

$$\Rightarrow X(S)\left[M_2 s^2 + (B_2 + B)s + K\right] - X_1(S)(Bs + K) = F(S)$$

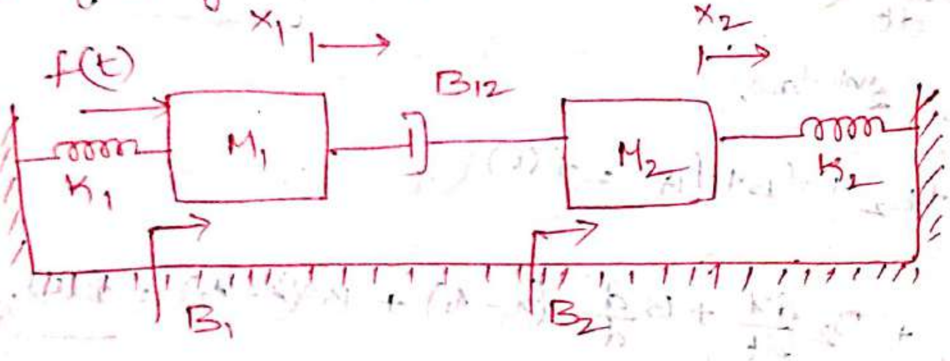
$$\Rightarrow X(S)\left[M_2 s^2 + (B + B_2)s + K\right] - \frac{(Bs + K)^2}{M_1 s^2 + (B_1 + B)s + (K_1 + K)} X(S) = F(S).$$

$$\Rightarrow X(S)\left(\frac{\left[M_1 s^2+(B_1+B)s+(K_1+K)\right]\left[M_2 s^2+(B_2+B)s+K\right]-(Bs+K)^2}{M_1 s^2+(B_1+B)s+(K_1+K)}\right) = F(S),$$

$$\therefore \frac{X(S)}{F(S)} = \frac{M_1 s^2+(B_1+B)s+(K_1+K)}{\left[M_1 s^2+(B_1+B)s+(K_1+K)\right]\left[M_2 s^2+(B_2+B)s+K\right]-(Bs+K)^2}$$

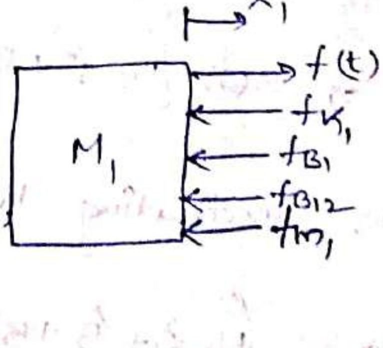
The Equations (A) and (B) are known as Differential Equations of a given system.

(2) Determine the transfer function $\frac{X_1(S)}{F(S)}$ and $\frac{X_2(S)}{F(S)}$ for the system given below,



Sol: given system, consists of two masses and so there by two differential equations will be obtained.

for Mass-1, the free body diagram will be



Now, $f_{m_1} = M_1 \frac{d^2x_1}{dt^2}$; $f_{B_1} = B_1 \frac{dx_1}{dt}$

$f_{B_{12}} = B_{12} \frac{d(x_1-x_2)}{dt}$, $f_{K_1} = K_1 x_1$

Now, By Newtons second law,

$$f_{m_1} + f_{K_1} + f_{B_1} + f_{B_{12}} = f(t).$$

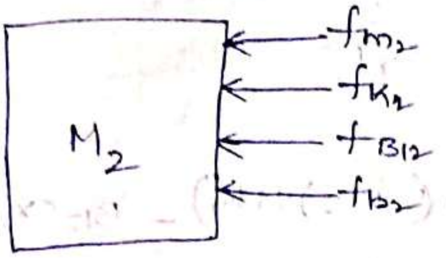
$$\Rightarrow M_1 \frac{d^2x_1}{dt^2} + B_1\frac{dx_1}{dt} + B_{12}\frac{d(x_1-x_2)}{dt} + K_1 x_1 = f(t) \quad \text{(A)}$$

Now apply the laplace transform for the above equation,

$$\Rightarrow M_1 s^2 X_1(s) + B_1 s X_1(s) + B_{12} s X_1(s) - B_{12} X_2(s) s + K_1 X_1(s) = F(s)$$

$$\Rightarrow X_1(s)\left[M_1 s^2 + B_1 s + B_{12} s + K_1\right] - B_{12} X_2(s) s = F(s) \quad \text{--- (1)}$$

Now for Mass-2, the force body diagram,



$$\therefore f_{m_2} = M_2 \frac{d^2 x_2}{dt^2} \; ; \; f_{b_2} = B_2 \frac{dx_2}{dt}$$

$$f_{B_{12}} = B_{12} \frac{d(x_2 - x_1)}{dt} \; ; \; f_{K_2} = K_2 X_2$$

By Newtons second law we have,

$$\therefore f_{m_2} + f_{b_2} + f_{B_{12}} + f_{K_2} = 0$$

$$\Rightarrow M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{dx_2}{dt} + B_{12} \frac{d(x_2 - x_1)}{dt} + K_2 X_2 = 0$$

By applying the laplace transform we get,

$$\Rightarrow M_2 s^2 X_2(s) + B_2 s X_2(s) + B_{12} s\left[X_2(s) - X_1(s)\right] + K_2 X_2(s) = 0$$

$$\Rightarrow X_2(s)\left[M_2 s^2 + B_2 s + B_{12} s + K_2\right] - B_{12} s X_1(s) = 0$$

$$\therefore X_2(s) = \frac{B_{12} s X_1(s)}{M_2 s^2 + B_2 s + B_{12} s + K_2} \quad \text{--- (2)}$$

By substituting eq. (2) in (1) we get

$$\therefore X_1(s)\left[M_1 s^2 + (B_1 + B_{12}) s + K_1\right] - \frac{(B_{12} s)^2 X_1(s)}{\left[M_2 s^2 + (B_2 + B_{12}) s + K_2\right]} = F(s)$$

$$\Rightarrow X_1(s) \frac{\left[M_1 s^2 + (B_1 + B_{12}) s + K_1\right]\left[M_2 s^2 + (B_2 + B_{12}) s + K_2\right] - (B_{12} s)^2}{\left[M_2 s^2 + (B_2 + B_{12}) s + K_2\right]} = F(s)$$

$$\therefore \frac{X_1(S)}{F(S)} = \frac{\left[M_2 S^2 + (B_2 + B_{12})S + K_2\right]}{\left[M_1 S^2 + (B_1 + B_{12})S + K_1\right]\left(M_2 S^2 + (B_2 + B_{12})S + K_2\right) - (B_{12}S)^2}$$

Similarly, from equation (2),

$$\Rightarrow X_1(S) = \frac{\left[M_2 S^2 + (B_2 + B_{12})S + K_2\right] X_2(S)}{B_{12}S} \quad \text{——— (3)}$$

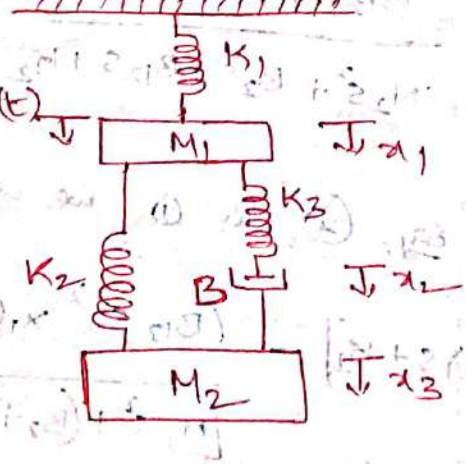
Now, Substitute eq (3) in (1) we get,

$$(1) \Rightarrow \frac{\left(M_2 S^2 + (B_2 + B_{12})S + K_2\right) X_2(S)}{B_{12}S}\left(M_1 S^2 + (B_1 + B_{12})S + K_1\right) - B_{12}S X_2(S) = F(S)$$

$$\Rightarrow X_2(S)\left[\frac{\left(M_2 S^2 + (B_2 + B_{12})S + K_2\right)\left(M_1 S^2 + (B_1 + B_{12})S + K_1\right) - (B_{12}S)^2}{B_{12}S}\right] = F(S)$$

$$\Rightarrow \frac{X_2(S)}{F(S)} = \frac{B_{12}S}{\left(M_2 S^2 + (B_2 + B_{12})S + K_2\right)\left(M_1 S^2 + (B_1 + B_{12})S + K_1\right) - (B_{12}S)^2}$$

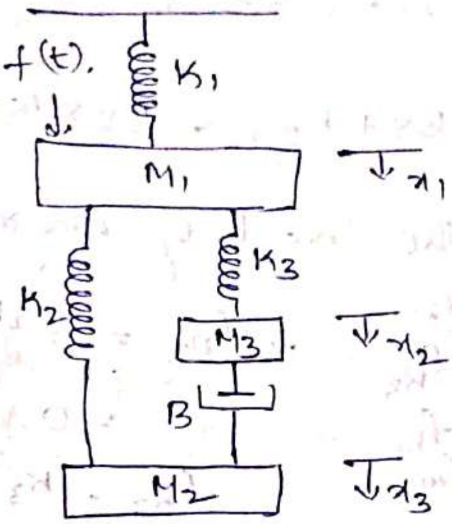
(3) Determine the differential equations ~~and the transfer function~~ from the Mechanical system given below,



[∵ element కి element కి మధ్య compulsory ఒక Mass ఉండాలి]

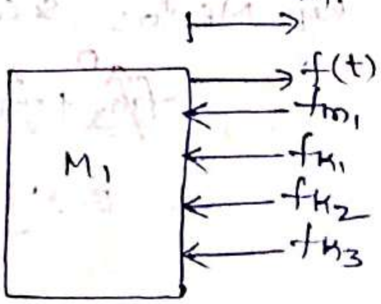
Note. No two elements should connect in series. ఒకవేళ రెండు series లో Connect అయినట్లు అయితే మధ్య ఒక Zero value గల Mass element ఉందని గా Assume చేసుకుని problem solve చేయాలి

Now, the assumed mechanical system is,



$[\because M_3 = 0]$

Now, for Mass-1 the force body diagram will be,



$\therefore f_{m_1} = M_1 \frac{d^2 x_1}{dt^2}$; $f_{K_1} = K_1 x_1$

$f_{K_2} = K_2 (x_1 - x_3)$; $f_{K_3} = K_3 (x_1 - x_2)$

$\therefore$ By Newtons second law,

$f_{m_1} + f_{K_1} + f_{K_2} + f_{K_3} = f(t)$

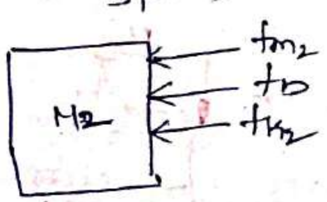
$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + K_1 x_1 + K_2 (x_1 - x_3) + K_3 (x_1 - x_2) = F(t) \quad \text{——— (A)}$$

By applying laplace transform we get,

$$\Rightarrow M_1 s^2 X_1(S) + K_1 X_1(S) + K_2 [X_1(S) - X_3(S)] + K_3 [X_1(S) - X_2(S)] = F(S).$$

$$\Rightarrow X_1(S) [M_1 s^2 + K_1 + K_2 + K_3] - K_2 X_3(S) - K_3 X_2(S) = F(S). \quad \text{——— (1)}$$

Similarly, for, Mass-2, force body diagram will be,



$\therefore f_{m_2} = M_2 \frac{d^2 x_3}{dt^2}$; $f_b = B \frac{d(x_3 - x_2)}{dt}$

$f_{K_2} = K_2 (x_3 - x_1)$

$\therefore$ By Newtons law, $f_{m_2} + f_b + f_{K_2} = 0$

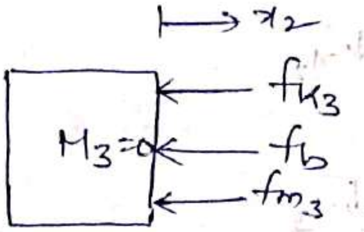
$$\Rightarrow M_2 \frac{d^2 x_3}{dt^2} + B \frac{d(x_3 - x_2)}{dt} + K_2 (x_3 - x_1) = 0 \quad \text{——— (B)}$$

Apply laplace transform we get,

$$\Rightarrow M_2 s^2 X_3(S) + BS\left[X_3(S) - X_2(S)\right] + K_2\left[X_3(S) - X_1(S)\right] = 0$$

$$\Rightarrow X_3(S)\left[M_2 s^2 + BS + K_2\right] - BSX_2(S) - K_2 X_1(S) = 0 \quad \text{——} (2)$$

Now, for Mass-3, the free body diagram will be,



$$\therefore f_{m3} = M_3 \frac{d^2 x_2}{dt^2} = 0 \quad ; \quad f_b = B\frac{d(x_2 - x_3)}{dt}$$

$$f_{k3} = K_3(x_2 - x_1)$$

By Newtons second law,

$$\cancel{f_{m3}}^{0} + f_b + f_{k3} = 0$$

$$\Rightarrow B\frac{d(x_2 - x_3)}{dt} + K_3(x_2 - x_1) = 0 \quad \text{——} (C)$$

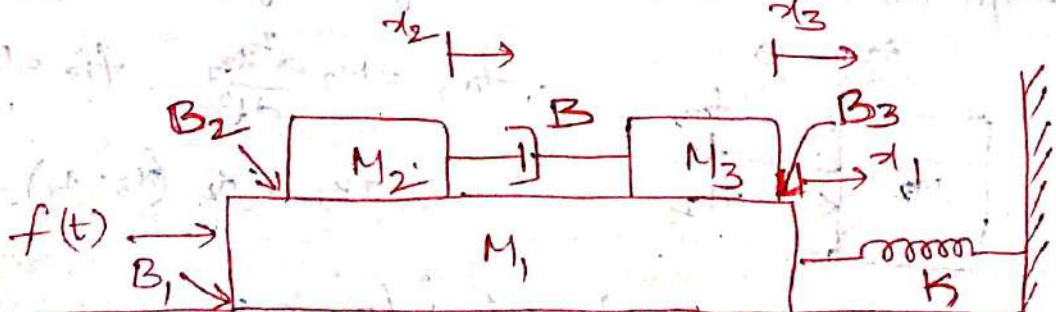
By applying laplace transform to above equation,

$$\Rightarrow BSX_2(S) - BSX_3(S) + K_3X_2(S) - K_3X_1(S) = 0$$

$$\Rightarrow X_2(S)(BS + K_3) - BSX_3(S) - K_3X_1(S) = 0 \quad \text{——} (3)$$

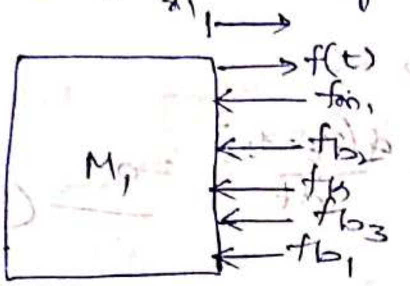
The Equations (A), (B), (C) are known as Differential Equations of the Mechanical Translational System

(4) Determine the Differential equations for the given Mechanical Translational System



Sol. In the given translational system there are 3-masses, so, 3 free body diagrams and 3-differential equations will be obtained.

for Mass-1 free body diagram will be,



$\therefore f_{m_1} = M_1 \frac{d^2 x_1}{dt^2}$; $f_{b_1} = B_1 \frac{dx_1}{dt}$

$f_{b_2} = B_2 \frac{d(x_1 - x_2)}{dt}$; $f_{b_3} = B_3 \frac{d(x_1 - x_3)}{dt}$

$f_k = K . x_1$

By Newtons second law,

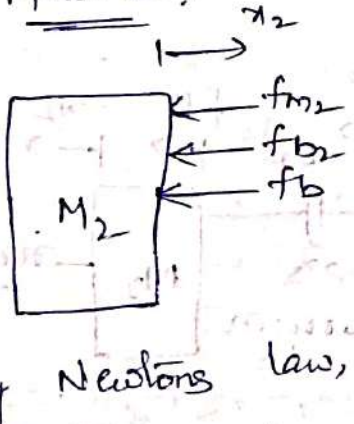
$$f_{m_1} + f_k + f_{b_1} + f_{b_2} + f_{b_3} = f(t)$$

$$\Rightarrow M_1 \frac{d^2 x_1}{dt^2} + K . x_1 + B_1 \frac{dx_1}{dt} + B_2 \frac{d(x_1 - x_2)}{dt} + B_3 \frac{d(x_1 - x_3)}{dt} = f(t)$$

By applying laplace transform,

$$M_1 s^2 X_1(S) + K X_1(S) + B_1 S X_1(S) + B_2 S [X_1(S) - X_2(S)] + B_3 S [X_1(S) - X_3(S)] = F(S) \quad (1)$$

for Mass-2 :-



$\therefore f_{m_2} = M_2 \frac{d^2 x_2}{dt^2}$; $f_{b_2} = B_2 \frac{d(x_2 - x_1)}{dt}$

$f_b = B \frac{d(x_2 - x_3)}{dt}$

$\therefore$ By Newtons law,

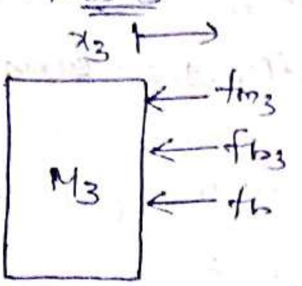
$$f_{m_2} + f_{b_2} + f_b = 0$$

$$\Rightarrow M_2 \frac{d^2 x_2}{dt^2} + B_2 \frac{d(x_2 - x_1)}{dt} + B \frac{d(x_2 - x_3)}{dt} = 0$$

Apply laplace transform.

$$\Rightarrow M_2 s^2 X_2(S) + B_2 S X_2(S) - B_2 S X_1(S) + BS X_2(S) - BS X_3(S) = 0 \quad (2)$$

for Mass-3.



$\therefore f_{m3} = M_3 \dfrac{d^2x_3}{dt^2}$; $f_{b3} = B_3 \dfrac{d(x_3-x_1)}{dt}$

$f_b = B \dfrac{d(x_3-x_2)}{dt}$

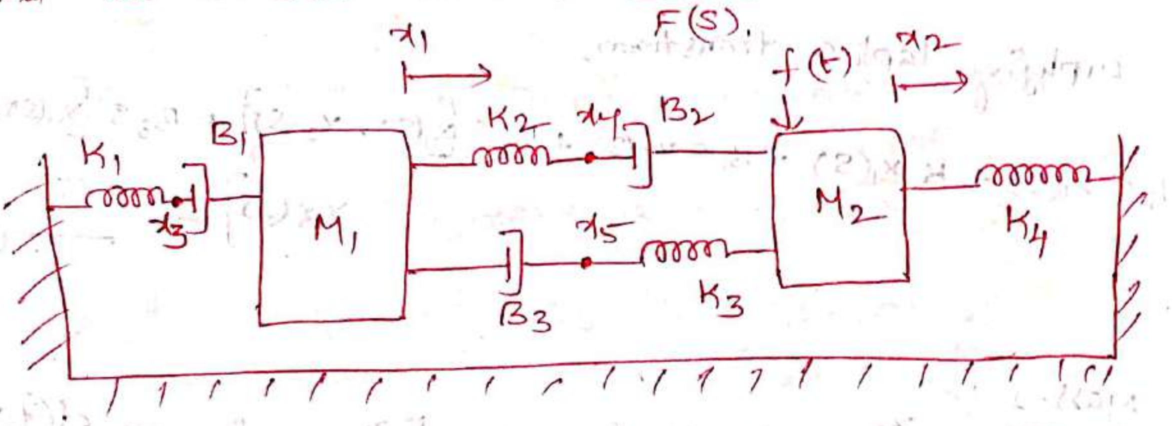
$\therefore$ By Newtons law ; $f_{m3}+f_{b3}+f_b = 0$

$$\Rightarrow M_3 \frac{d^2x_3}{dt^2} + B_3 \frac{d(x_3-x_1)}{dt} + B\frac{d(x_3-x_2)}{dt} = 0 \quad \text{—— (3)}$$

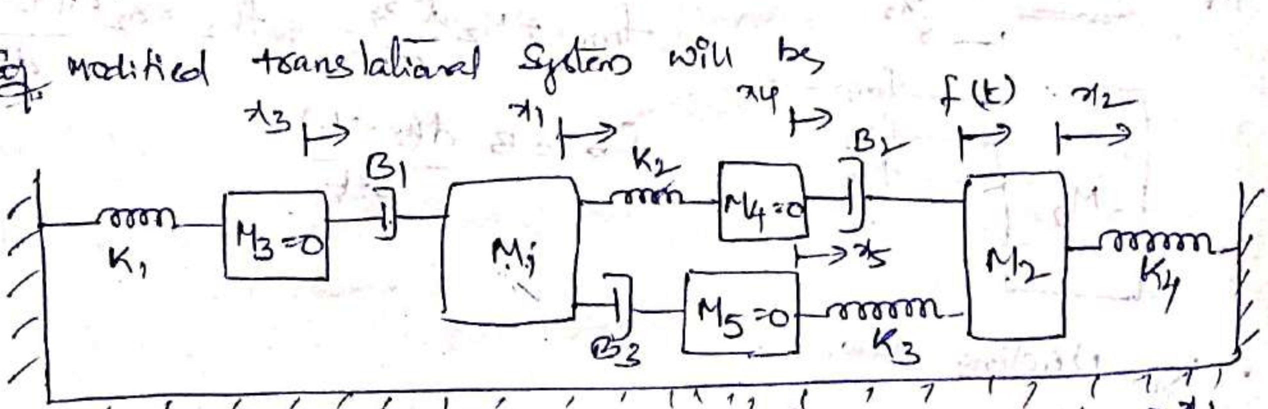
Apply laplace transform,

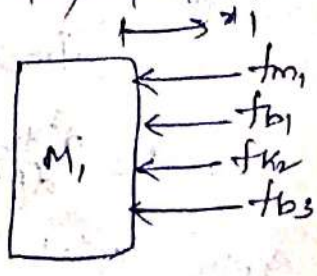
$$M_3 s^2 X_3(s) + B_3 s\,[X_3(s) - X_1(s)] + Bs\,[X_3(s) - X_2(s)] = 0$$

(5) For the mechanical translational system shown below find the transfer function $\dfrac{X_2(s)}{F(s)}$.



Sol. modified translational system will be,





for Mass-1,

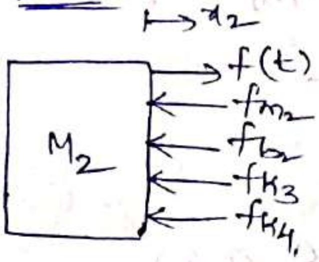
$\therefore f_{m1} + f_{b3} + f_{b1} + f_{k2} = 0$

$$\Rightarrow M_1 \frac{d^2x_1}{dt^2} + B_3 \frac{d(x_1-x_5)}{dt} + B_1\frac{d(x_1-x_3)}{dt} + K_2\cdot(x_1-x_4) = 0 \quad \text{—— (A)}$$

$\Rightarrow M_1 s^2 X_1(s) + B_3 s[X_1(s) - X_5(s)] + B_1 s[X_1(s) - X_3(s)] + K_2[X_1(s) - X_4(s)] = 0$

$\Rightarrow X_1(s)[M_1 s^2 + B_3 s + B_1 s + K_2] - B_3 s X_5(s) - B_1 s X_3(s) - K_2 X_4(s) = 0$ —— (1)

for Mass-2



$\therefore f_{m_2} = M_2 \dfrac{d^2 x_2}{dt^2}$; $f_{b_2} = B_2 \dfrac{d(x_2 - x_4)}{dt}$

$f_{k_3} = K_3(x_2 - x_5)$; $f_{k_4} = K_4 x_2$

$\therefore f_{m_2} + f_{b_2} + f_{k_3} + f_{k_4} = f(t)$.

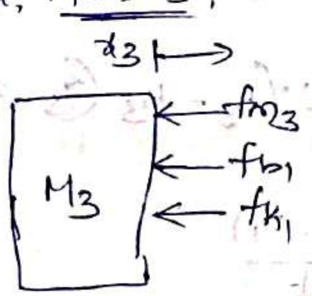
$\therefore M_2 \dfrac{d^2 x_2}{dt^2} + B_2 \dfrac{d(x_2 - x_4)}{dt} + K_3(x_2 - x_5) + K_4 x_2 = f(t)$. —— (B)

Applying laplace transform to above Equation we get

$\Rightarrow M_2 s^2 X_2(s) + B_2 s[X_2(s) - X_4(s)] + K_3[X_2(s) - X_5(s)] + K_4[X_2(s)] = F(s)$

$\Rightarrow X_2(s)[M_2 s^2 + B_2 s + K_3 + K_4] - B_2 s X_4(s) - K_3 X_5(s) = F(s)$. —— (2)

for, Mass-3 :-



$f_{M_3} = M_3 \dfrac{d^2 x_3}{dt^2} = 0$; $f_{b_1} = B_1 \dfrac{d(x_3 - x_1)}{dt}$

$f_{k_1} = K_1 x_3$.

$\therefore f_{b_1} + f_{k_1} = 0$

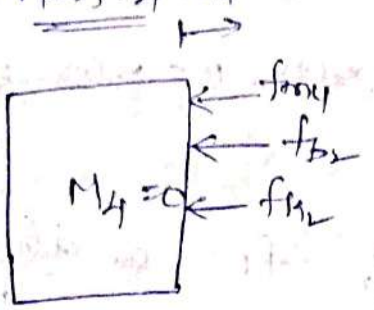
$\Rightarrow B_1 \dfrac{d(x_3 - x_1)}{dt} + K_1 x_3 = 0$ —— (C)

Apply laplace transform.

$B_1 s X_3(s) - B_1 s X_1(s) + K_1 X_3(s) = 0$

$\Rightarrow X_3(s)[B_1 s + K_1] - B_1 s X_1(s) = 0$ —— (3)

for Mass-4 :



$$f_{m_4} = M_4 \frac{d^2 x_4}{dt^2} = 0 \; ; \; f_{b_2} = B_2 \frac{d(x_4 - x_2)}{dt}$$

$$f_{k_2} = K_2 (x_4 - x_1)$$

$$\therefore f_{b_2} + f_{k_2} = 0$$

$$\Rightarrow B_2 \frac{d(x_4 - x_2)}{dt} + K_2(x_4 - x_1) = 0 \quad \text{——— (A) (D)}$$

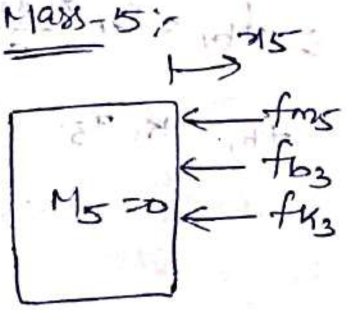
Applying laplace transform we get,

$$\Rightarrow B_2 s X_4(S) - B_2 s X_2(S) + K_2 X_4(S) - K_2 X_1(S) = 0$$

$$\Rightarrow X_4(S) \left[B_2 s + K_2\right] - B_2 s X_2(S) - K_2 X_1(S) = 0$$

$$\Rightarrow \boxed{X_4(S) = \frac{B_2(S) X_2(S) + K_2 X_1(S)}{B_2 s + K_2}} \quad \text{——— (A)}$$

for Mass-5 :



$$f_{m_5} = M_5 \frac{d^2 x_5}{dt^2} = 0 \; ; \; f_{k_3} = K_3 (x_5 - x_2)$$

$$f_{b_3} = B_3 \frac{d(x_5 - x_1)}{dt}$$

$$\therefore f_{k_3} + f_{b_3} = 0$$

$$\Rightarrow K_3(x_5 - x_2) + B_3 \frac{d(x_5 - x_1)}{dt} = 0 \quad \text{——— (B) (E)}$$

$$\Rightarrow K_3 X_5(S) - K_3 X_2(S) + B_3 s X_5(S) - B_3 s X_1(S) = 0$$

$$\Rightarrow X_5(S)\left[K_3 + B_3 s\right] - K_3 X_2(S) - B_3 s X_1(S) = 0$$

$$\Rightarrow \boxed{X_5(S) = \frac{K_3 X_2(S) + B_3 s X_1(S)}{K_3 + B_3 s}} \quad \text{——— (5)}$$

when displacements are Replaced by velocities then,

$$\therefore \frac{d^2x}{dt^2} = \frac{dv}{dt}$$

$$\frac{dx}{dt} = v$$

$$x = \int v \, dt$$

$$\therefore (A) \Rightarrow M_1 \frac{dv}{dt} + B_3 \frac{d(v_1 - v_5)}{\cancel{dt}} + B_1 \frac{d(v_1 - v_3)}{\cancel{dt}} + K_2 \int (v_1 - v_4)\, dt = 0$$

$$(B) \Rightarrow M_2 \frac{dv_2}{dt} + B_2 (v_2 - v_4) + K_3 \int (v_2 - v_5)\, dt + K_4 \int v_2 \, dt = f(t)$$

$$(C) \Rightarrow B_1 (v_3 - v_1) + K_1 \int v_3 \, dt = 0$$

$$(D) \Rightarrow B_2 (v_4 - v_2) + K_2 \int (v_4 - v_1)\, dt = 0$$

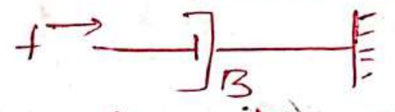
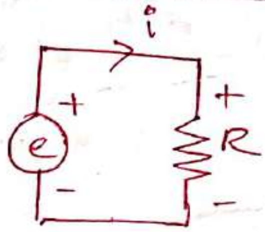
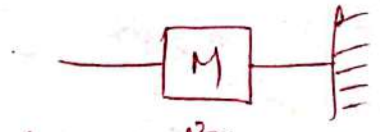
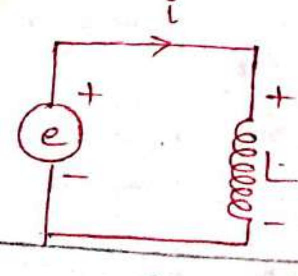
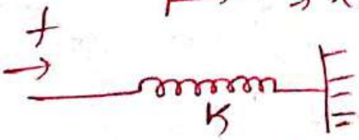
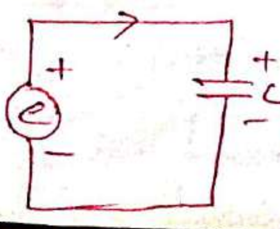
$$(E) \Rightarrow K_3 \int (v_5 - v_2)\, dt + B_3 (v_5 - v_1) = 0$$

* Electrical Analogous of Mechanical Translational System:-

The three basic elements Mass, Dashpot and Spring in Modelling of Mechanical Translational System are Analogous (identical) to Resistance, inductance and capacitance of electrical system.

The input force in Mechanical system is analogous to voltage (or) current. Similarly, the output velocity is identical to either voltage (or) current in an electrical system since, electrical system has two types of inputs, voltage and current, so, there are two analogies i, force-voltage Analogy (2) Force-Current Analogy.

* Force - Voltage Analogy :-

Mechanical System	Electrical system
Input :- Force Output :- Velocity	Input :- Voltage Source output :- Current through element
$\mapsto x$ $\mapsto v = dx/dt$  $f_B = B \cdot \frac{dx}{dt} = B \cdot v$	 $e = Ri$
$\mapsto x$ $\quad \frac{d^2x}{dt^2} = \frac{dv}{dt}$  $f_m = M \frac{d^2x}{dt^2} = M \cdot \frac{dv}{dt}$	 $e = L \frac{di}{dt}$
$\mapsto x \Rightarrow x = \int v\,dt$  $f_K = K \cdot x = K \cdot \int v\,dt$	 $e = \frac{1}{C}\int i\,dt$

Procedure :-

1. In an Electrical System, the elements in Series will have same Current, like wise in Mechanical Systems, the elements having same velocity are said to be in Series.

2. The elements having same velocity in Mechanical System Should have analogous same current in Electrical Analogous System

3. Each Node in Mechanical System Corresponds to closed loop in Electrical System. A Mass is Considered as a Node.

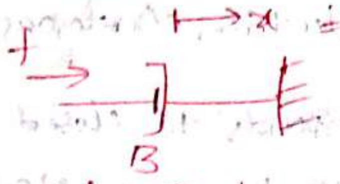
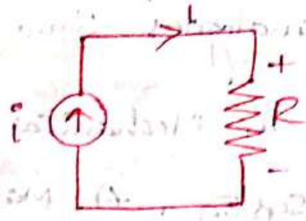
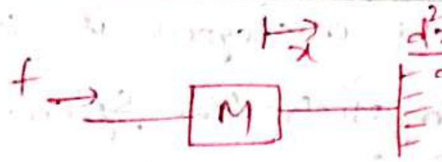
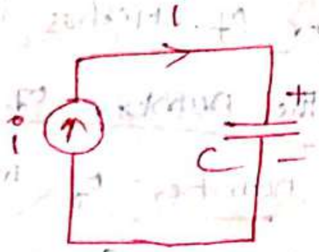
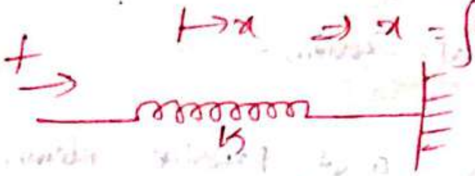
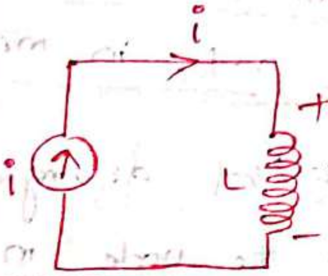
4. The number of Meshes in Electrical analogous is same as that of the number of nodes in Mechanical System. Hence, the number of mesh currents and System equations will be same as that of the number of velocities of nodes in mechanical System.

5. The mechanical driving sources and passive elements Connected to the node in mechanical System should be Represented by analogous elements in a closed loop in a analogous electrical System.

6. The element Connected between two masses in the Mechanical System is represented as a Common element between two meshes in electrical analogous System.

NOTE :- force - Voltage Analogy Goes Mesh Analysis Method.
force - Current Analogy Goes Nodal Analysis Method.

* Force - Current Analogy :-

Mechanical System	Electrical System
Input :- Force output :- Velocity	Input :- Current Source output :- Voltage across element
 x, $\frac{dx}{dt} = v$ $f_B = B\cdot v$	 $\Rightarrow i = \frac{1}{R} v$
 $\frac{d^2x}{dt^2} = \frac{dv}{dt}$ $f_m = M\frac{dv}{dt}$	 $\Rightarrow i = C\frac{dv}{dt}$
 $\Rightarrow x = \int v\,dt$ $f_K = Kx = K\int v\,dt$	 $\Rightarrow i = \frac{1}{L}\int v\,dt$

Procedure :-

1. The Electrical systems, element in parallel will have same voltage, likewise in mechanical systems, the elements having same force are said to be in parallel.

2. The elements having same velocity in mechanical system should have analogous same voltage in Electrical analogous system.

3. Each Node in mechanical system corresponds to a node in Electrical system. A mass is considered as a Node.

4. The number of nodes in Electrical system is same as that of number of nodes in mechanical system. Hence, the number of node voltages and system equations will be same as that of the number of velocities of masses in mechanical system.

5. The mechanical driving sources and passive elements connected to the node in mechanical system should be represented by analogous elements connected to a node in a Electrical system.

6. The element connected between two nodes in the mechanical system is represented as a common element between two nodes in Electrical analogous system.

* for Force - Voltage system :-

$f(t) \rightarrow e(t)$

$B \rightarrow R$

$M \rightarrow L$

$K \rightarrow \frac{1}{C}$

[Common ga ఉన్నటి elements ఉంటే Loops ki Common గా ఉన్నటిలో చూసుకోవాలి.]

* For force - Current system :-

$f(t) \rightarrow i(t)$

$B \rightarrow \frac{1}{R}$

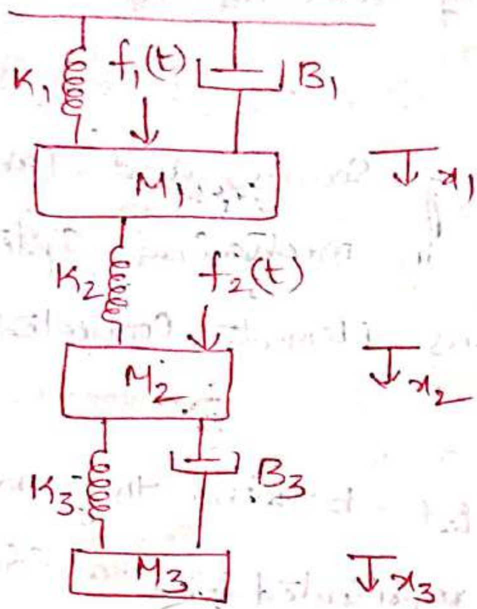
$K \rightarrow \frac{1}{L}$

$M \rightarrow C$

[Common గా ఉన్నటి elements ఉంటే Nodes కి Common గా ఉన్నటిలో చూసుకోవాలి]

* problem :-

① Draw the force-voltage and force-current Analogy ckts for the given System.



number of masses = number of loops.

$f \to V$
$B \to R$
$M \to L$
$K \to C$

Sol. The differential Equations are,

$$M_1 \frac{d^2x_1}{dt^2} + K_1 x_1 + B_1 \frac{dx_1}{dt} + K_2(x_1 - x_2) = f_1(t) \quad \text{—(A)}$$

$$M_2 \frac{d^2x_2}{dt^2} + K_2(x_2 - x_1) + K_3(x_2 - x_3) + B_3 \frac{d(x_2 - x_3)}{dt} = f_2(t) \quad \text{—(B)}$$

$$M_3 \frac{d^2x_3}{dt^2} + K_3(x_3 - x_2) + B_3 \frac{d(x_3 - x_2)}{dt} = 0 \quad \text{—(C)}$$

$$\left[\because \frac{d^2x}{dt^2} = \frac{dv}{dt} \ ; \ \frac{dx}{dt} = v \ ; \ x = \int v\,dt\right]$$

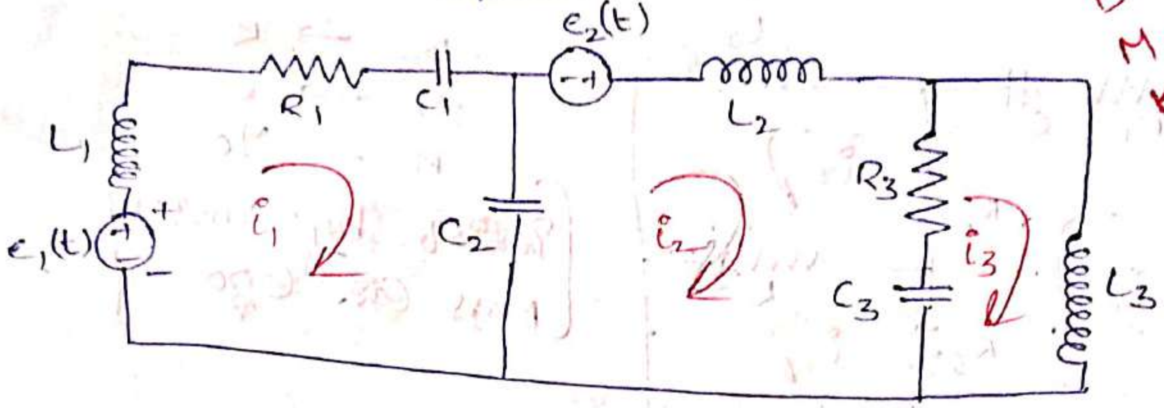
$$\Rightarrow M_1 \frac{dv_1}{dt} + K_1 \int v_1\,dt + B_1 v + K_2 \int (v_1 - v_2)\,dt = f_1(t) \quad \text{—(1)}$$

$$\Rightarrow M_2 \frac{dv_2}{dt} + K_2 \int (v_2 - v_1)\,dt + K_3 \int (v_2 - v_3)\,dt + B_3(v_2 - v_3) = f_2(t) \quad \text{—(2)}$$

$$\Rightarrow M_3 \frac{dv_3}{dt} + K_3 \int (v_3 - v_2)\,dt + B_3(v_3 - v_2) = 0 \quad \text{—(3)}$$

force-voltage Analogy ckt :

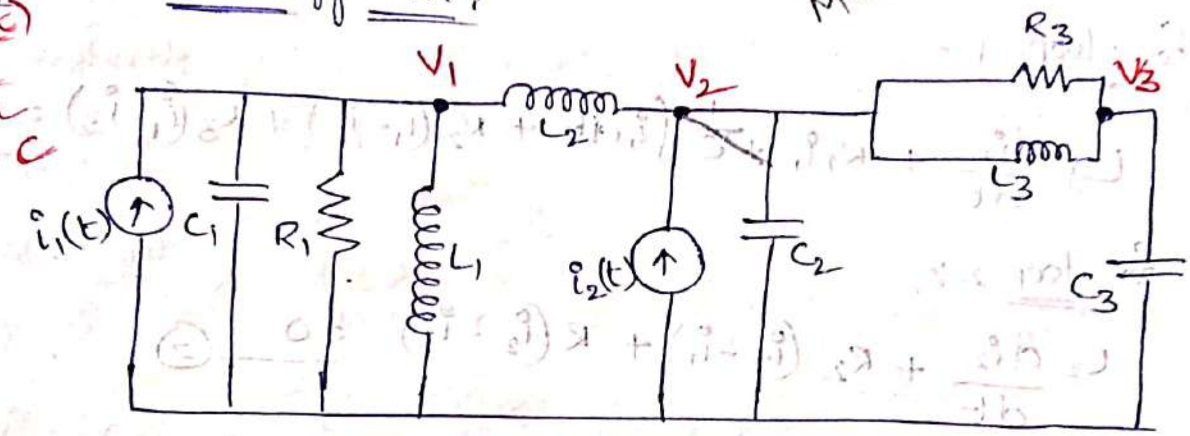
$B \rightarrow R$
$M \rightarrow L$
$K \rightarrow C$
$f(t) \rightarrow e(t)$



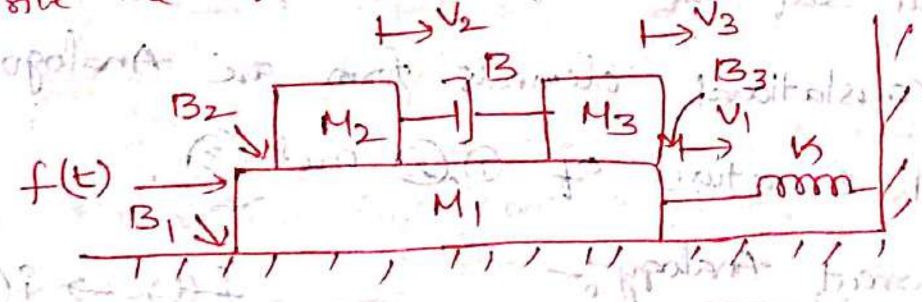
force-current Analogy ckt :

Mass = Nodes

$f(t) \rightarrow i(t)$
$B \rightarrow R$
$K \rightarrow L$
$M \rightarrow C$



(2) Determine the Differential equations for given system and draw the force voltage and force current Analogous ckts. Also write the equations related to them



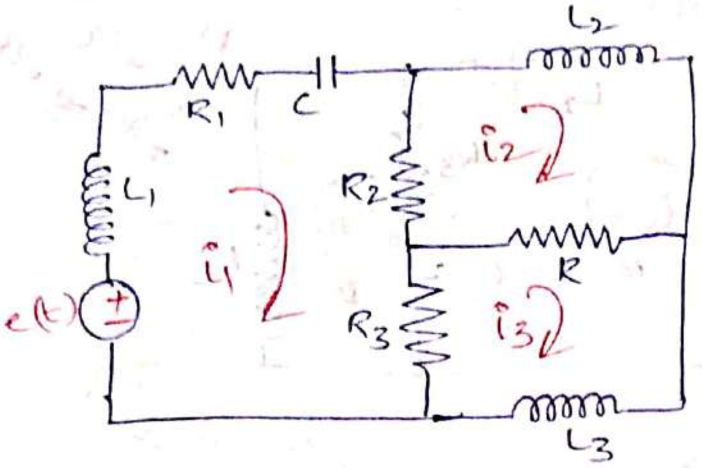
Sol:

$$M_1 \frac{dv_1}{dt} + K\int v_1 dt + B_1 v_1 + B_2(v_1 - v_2) + B_3(v_1 - v_3) = f(t) \quad \text{— (A)}$$

$$M_2 \frac{dv_2}{dt} + B_2(v_2 - v_1) + B(v_2 - v_3) = 0 \quad \text{— (B)}$$

$$M_3 \frac{dv_3}{dt} + B(v_3 - v_2) + B_3(v_3 - v_1) = 0 \quad \text{— (C)}$$

(ii) force - voltage Analogy :-



$f(t) \rightarrow e(t)$
$B \rightarrow R$
$M \rightarrow L$
$K \rightarrow 1/C$

[ఎన్ని loop ఎన్ని Mass లు ఉంటె]

for loop - 1 :-

$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C}\int i_1 dt + R_2(i_1 - i_2) + R_3(i_1 - i_3) = e(t) \quad (1)$$

for loop - 2 :-

$$L_2 \frac{di_2}{dt} + R_2(i_2 - i_1) + R(i_2 - i_3) = 0 \quad (2)$$

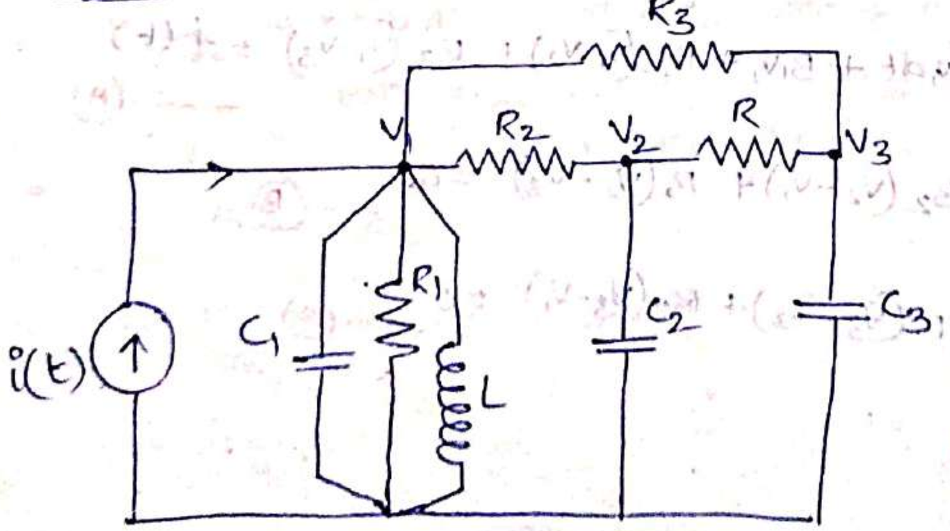
for loop - 3 :-

$$L_3 \frac{di_3}{dt} + R_3(i_3 - i_1) + R(i_3 - i_2) = 0 \quad (3)$$

Now, we can say the equations of (A), (B), (C) which are in Translational elements form are Analogous to Electrical equations of (1), (2) and (3).

for force - Current Analogy :-

$f(t) \rightarrow i(t)$
$B \rightarrow 1/R$
$K \rightarrow 1/L$
$M \rightarrow C$



[ఎన్ని Node ఎన్ని Mass లు ఉంటె]

for Node-1:

$$C_1 \frac{dV_1}{dt} + \frac{1}{R_1} V_1 + \frac{1}{L}\int V_1 dt + \frac{1}{R_2}(V_1 - V_2) + \frac{(V_1 - V_3)}{R_3} = i(t) \quad \text{--- (4)}$$

for Node-2:

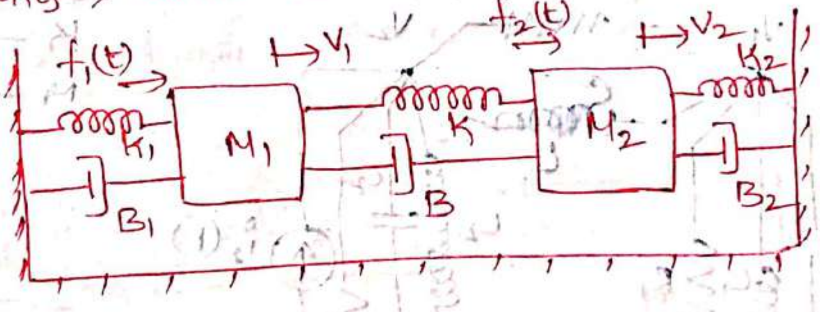
$$C_2 \frac{dV_2}{dt} + \frac{(V_2 - V_1)}{R_2} + \frac{(V_2 - V_3)}{R} = 0 \quad \text{--- (5)}$$

for Node-3:

$$C_3 \frac{dV_3}{dt} + \frac{(V_3 - V_1)}{R_3} + \frac{(V_3 - V_2)}{R} = 0 \quad \text{--- (6)}$$

∴ The equations (4), (5) and (6) are analogous to mechanical translational system equations of (A), (B) & (C) which are obtained by using force - Current Analogy.

(3) Write the Differential Equations of a given system. Also draw the force - voltage and force - Current analogy ckts, write the Relational equations for them.



Sol The differential Equations will be,

$$M_1 \frac{d^2 x_1}{dt^2} + K_1 x_1 + B_1 \frac{dx_1}{dt} + K (x_1 - x_2) + B \frac{d(x_1 - x_2)}{dt} = f_1(t)$$

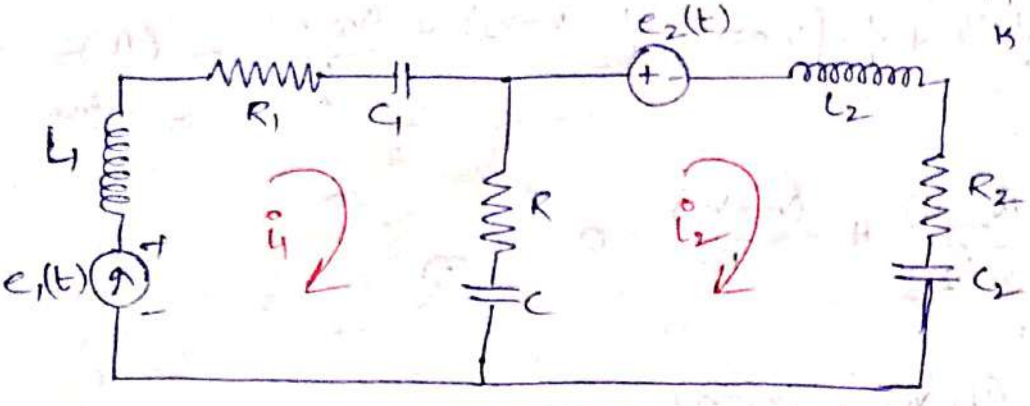
$$\Rightarrow M_1 \frac{dV_1}{dt} + K_1 \int V_1 dt + B_1 V_1 + K\int (V_1 - V_2) dt + B(V_1 - V_2) = f_1(t) \quad \text{--- (1)}$$

$$M_2 \frac{d^2 x_2}{dt^2} + K_2 x_2 + B_2 \frac{dx_2}{dt} + K (x_2 - x_1) + B \frac{d(x_2 - x_1)}{dt} = f_2(t)$$

$$\Rightarrow M_2 \frac{dV_2}{dt} + K_2 \int V_2 dt + B_2 V_2 + K\int (V_2 - V_1) dt + B(V_2 - V_1) = f_2(t) \quad \text{--- (2)}$$

for force-voltage Analogy :-

B → R
M → L
K → $\frac{1}{C}$



for loop-1 :-

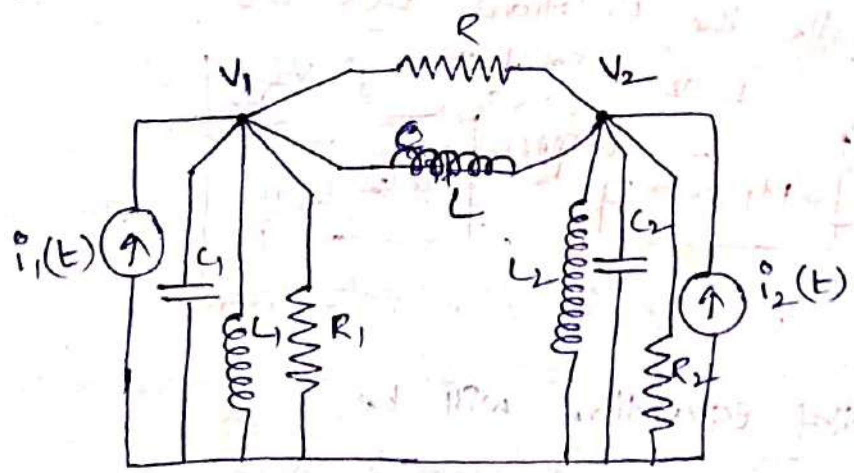
$$L_1 \frac{di_1}{dt} + R_1 i_1 + \frac{1}{C_1}\int i_1 dt + R(i_1 - i_2) + \frac{1}{C}\int (i_1 - i_2) dt = e_1(t)$$

for loop-2 :-

$$L_2 \frac{di_2}{dt} + R_2 i_2 + R(i_2 - i_1) + \frac{1}{C_2}\int i_2 dt + \frac{1}{C}\int (i_2 - i_1) dt = e_2(t)$$

for force current Analogy :-

f(t) = i(t)
B → 1/R
K → 1/L
M → C



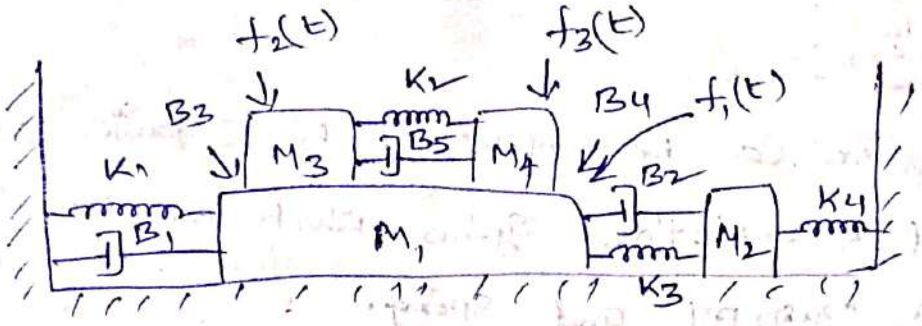
for Node-1 :-

$$C_1 \frac{dV_1}{dt} + \frac{V_1}{R_1} + \frac{1}{L_1}\int V_1 dt + \frac{(V_1 - V_2)}{R} + \frac{1}{L}\int (V_1 - V_2) dt = i_1(t)$$

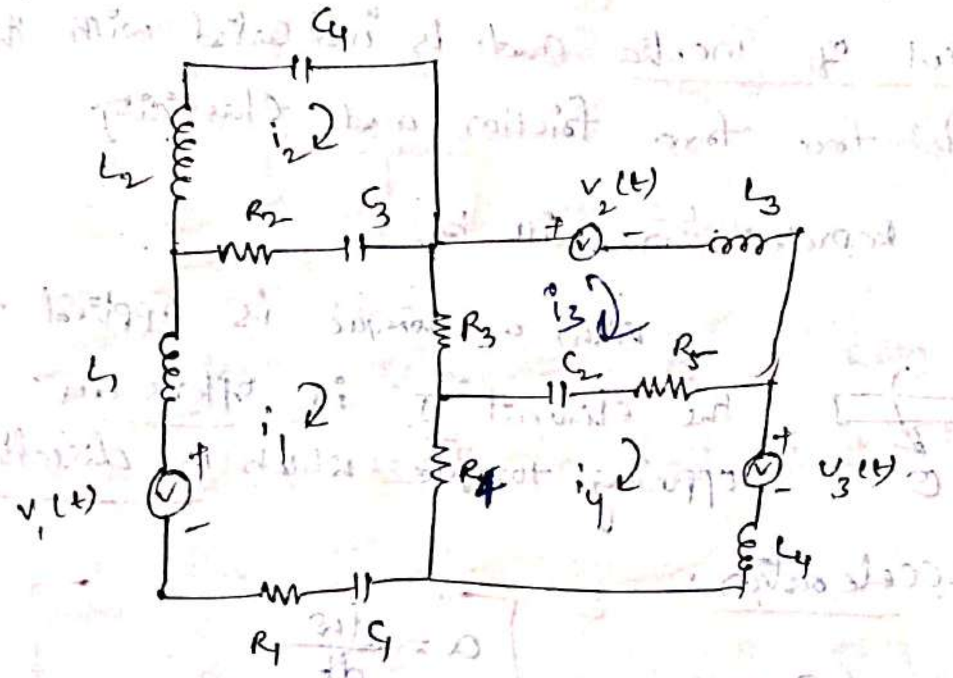
for ~~loop~~ Node-2 :-

$$C_2 \frac{dV_2}{dt} + \frac{V_2}{R_2} + \frac{V_2 - V_1}{R} + \frac{1}{L_2}\int V_2 dt + \frac{1}{L}\int (V_2 - V_1) dt = i_2(t)$$

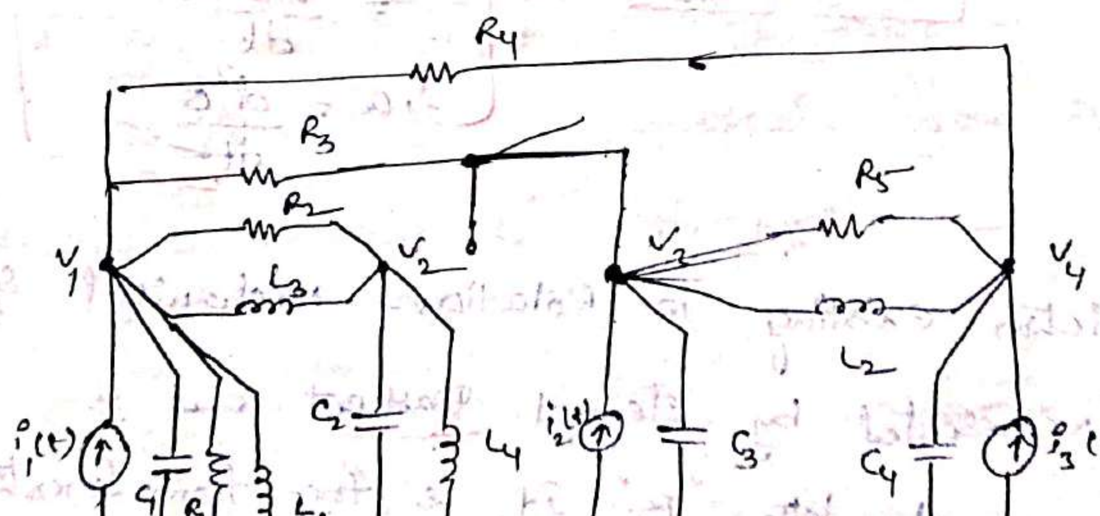
(4)



Sol: force-voltage.



force - current

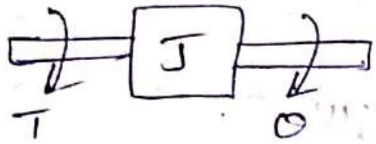


* Rotational System :-

Rotational System can be obtained by 3 basic elements as like Mechanical Translational System which are namely. Moment of Inertia, Dash pot and Spring.

The weight of Rotational System is represented by the element Moment of Inertia and is indicated with the letter 'J'. This is force from friction and Elasticity

The diagramatic Representation will be,



when a torque is applied to the element "J" it offers an opposing torque which is directly proportional to acceleration.

$\therefore \; T_J \propto a$

$$\Rightarrow \boxed{T_J = J \cdot \frac{d^2\theta}{dt^2}}$$

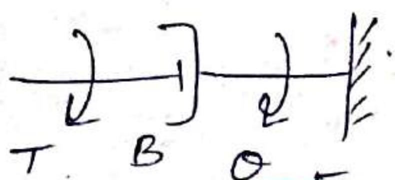
$a = \frac{d\omega}{dt}$

ω = angular velocity

$= \frac{d\theta}{dt}$

$\therefore a = \frac{d^2\theta}{dt^2}$

The friction existing in Rotational Mechanical System can be represented by element Dashpot and is indicated by the letter "B." It is force from Inertia and Elasticity. The diagramatic representation will be,

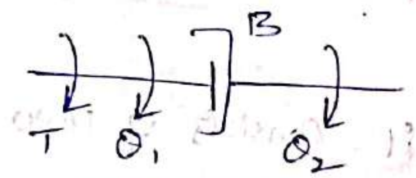


when a torque applied to Dashpot, it offers an opposing torque ~~force~~ which is directly proportional to Angular Velocity

$\therefore T_B \propto \omega$

$\Rightarrow \boxed{T_B = B \cdot \frac{d\theta}{dt}}$

when the Dashpot has the angular displacements on both sides, then the difference of both displacements can be considered.

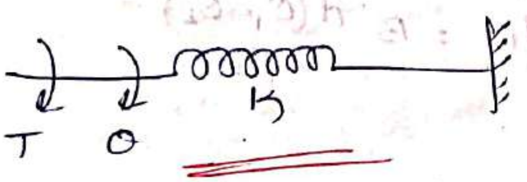


$\Rightarrow T_B \propto \omega$

$\Rightarrow \boxed{T_B = B \cdot \frac{d(\theta_1 - \theta_2)}{dt}}$

The Elastic deformation of a Rotational System can be Represented as the element spring and is indicated with a letter K.

The diagramatic Representation will be

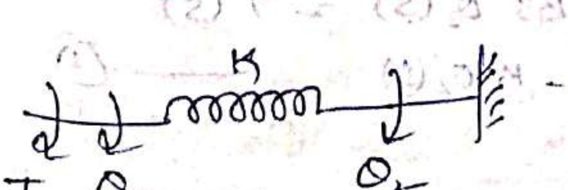


when a torque is applied $\Rightarrow$ to spring it offers an opposing torque which is directly proportional to displacement.

$T_K \propto \theta$

$\Rightarrow \boxed{T_K = K \cdot \theta}$

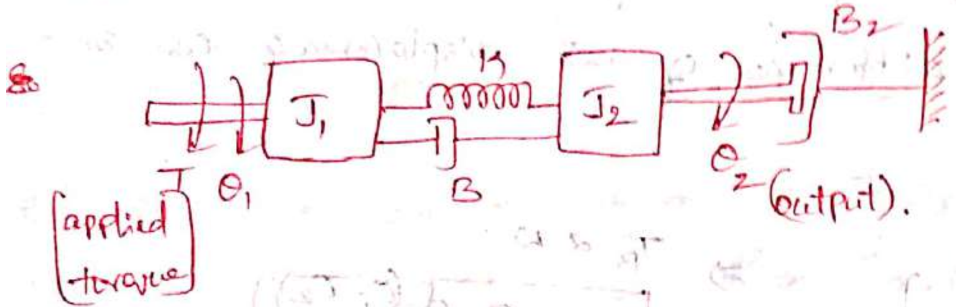
when a spring has two displacements on both sides then difference of displacements will be Considered.



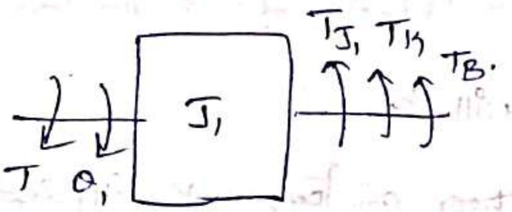
$\Rightarrow \boxed{T_K = K \cdot (\theta_1 - \theta_2)}$

* Problems :-

① Write the differential equations governing the system shown below and determine the transfer function $\frac{\theta_2(S)}{T(S)}$.



Sol. In the given Rotational System, it consists of two Inertia's. So, two differential equations will be existed for Moment of Inertia (J_1). The free body diagram will be



$$\Rightarrow T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_K = K \cdot \theta_1$$

$$T_B = B \frac{d(\theta_1 - \theta_2)}{dt}$$

By Newtons law,

$$T_{J_1} + T_K + T_B = T(t)$$

$$\Rightarrow J_1 \frac{d^2\theta_1}{dt^2} + \overset{K(\theta_1-\theta_2)}{\cancel{K\theta_1}} + B \frac{d(\theta_1 - \theta_2)}{dt} = T(t)$$

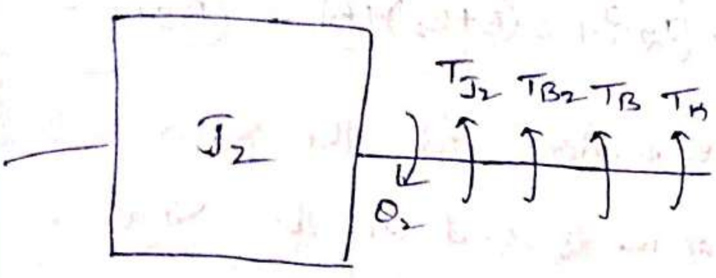
By applying laplace Transform we get,

$$\Rightarrow J_1 s^2 \theta_1(S) + K \cdot \theta_1(S) \overset{-K\cdot\theta_2(S)}{} + Bs\left[\theta_1(S) - \theta_2(S)\right] = T(S)$$

$$\Rightarrow \theta_1(S)\left[J_1 s^2 + K + Bs\right] - Bs\,\theta_2(S) - K\,\theta_2(S) = T(S) \quad \text{---(1)}$$

Similarly, for J_2 :-

free body diagram will be,



$$T_{J_2} = J_2 \frac{d^2\theta_2}{dt^2} \quad ; \quad T_B = B \cdot \frac{d(\theta_2 - \theta_1)}{dt}$$

$$T_{B_2} = B_2 \frac{d\theta_2}{dt} \quad ; \quad T_K = K \cdot (\theta_2 - \theta_1)$$

By Newtons law,

$$T_{J_2} + T_K + T_B + T_{B_2} = 0$$

$$\Rightarrow J_2 \frac{d^2\theta_2}{dt^2} + K \cdot (\theta_2 - \theta_1) + B \frac{d(\theta_2 - \theta_1)}{dt} + B_2 \frac{d\theta_2}{dt} = 0$$

$\Rightarrow$ By applying laplace Transform we get

$$\Rightarrow J_2 s^2 \theta_2(S) + K\left[\theta_2(S) - \theta_1(S)\right] + BS\left(\theta_2(S) - \theta_1(S)\right] + B_2 S\left[\theta_2(S)\right] = 0$$

$$\Rightarrow \theta_2(S)\left[J_2 s^2 + K + (B + B_2)s\right] - \theta_1(S)(BS + K) = 0 \quad \text{—— (2)}$$

from (2) $\Rightarrow$ $$\theta_1(S) = \frac{\left[J_2 s^2 + (B + B_2)s + K\right]\theta_2(S)}{BS + K} \quad \text{—— (3)}$$

Substituting (3) in (1) we get,

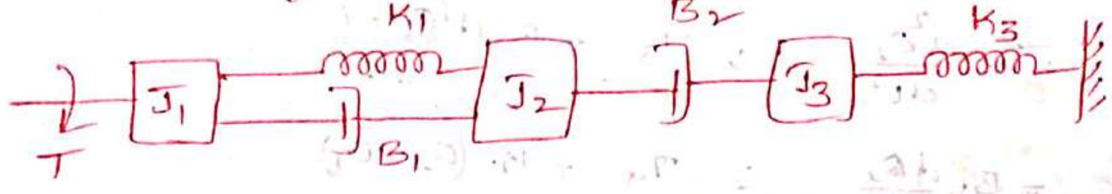
$$\Rightarrow \left[J_1 s^2 + BS + K\right] \frac{\left[J_2 s^2 + (B + B_2)s + K\right]\theta_2(S)}{BS + K} - (BS + K)\theta_2(S) = T(S)$$

$$\Rightarrow \left\{ \frac{\left[J_1 s^2 + K + BS\right]\left[J_2 s^2 + K + (B + B_2)s\right] - \left[BS + K\right]^2}{BS + K} \right\} \theta_2(S) = T(S).$$

∴ Now, the Required Transfer function is,

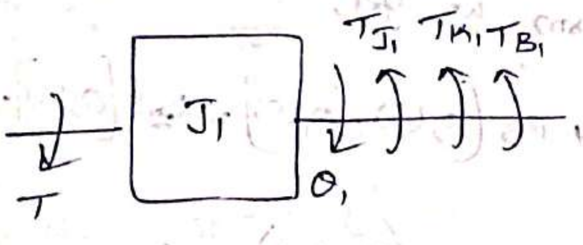
$$\frac{\theta_2(S)}{T(S)} = \frac{BS+K}{(J_1 S^2 + K + BS)\left[J_2 S^2 + S(B+B_2) + K\right] - (BS+K)^2}$$

② Write the differential equations for the given system for both angular displacements and angular velocities.



Sol. from the given Mechanical Rotational system we have 3 Inertias. So, 3 differential equations will be obtained (or) calculated.

for J_1 :-



$$\Rightarrow T_{J_1} = J_1 \frac{d^2\theta_1}{dt^2}$$

$$T_{K_1} = K_1 \cdot (\theta_1 - \theta_2)$$

$$T_{B_1} = B_1 \cdot \frac{d(\theta_1 - \theta_2)}{dt}$$

By Newtons law,

$$T_{J_1} + T_{K_1} + T_{B_1} = T(t)$$

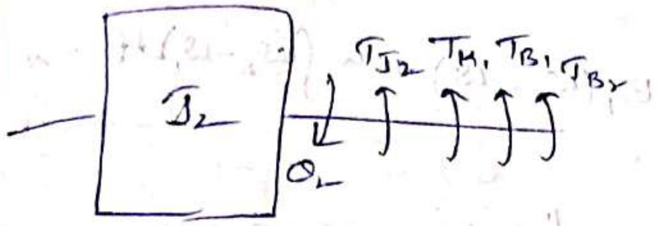
$$\Rightarrow J_1 \frac{d^2\theta_1}{dt^2} + K_1(\theta_1 - \theta_2) + B_1 \cdot \frac{d(\theta_1 - \theta_2)}{dt} = T(t)$$

By applying Laplace Transform we get,

$$\Rightarrow J_1 S^2 \theta_1(S) + K_1\left[\theta_1(S) - \theta_2(S)\right] + B_1 S \cdot \left[\theta_1(S) - \theta_2(S)\right] = T(S)$$

$$\Rightarrow \theta_1(S)\left[J_1 S^2 + B_1 S + K_1\right] - \theta_2(S)\left[B_1 S + K_1\right] = T(S). \quad \text{—①}$$

At J_2 :-



$$T_{J_2} = J_2 \frac{d^2\theta_2}{dt^2} ;$$

$$T_{K_1} = K_1 (\theta_2 - \theta_1)$$

$$T_{B_1} = B_1 \frac{d(\theta_2 - \theta_1)}{dt}$$

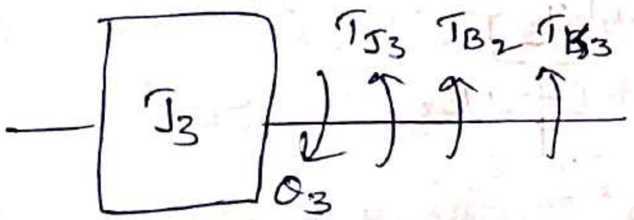
$$T_{B_2} = B_2 \frac{d(\theta_2 - \theta_3)}{dt}$$

By Newtons law,

$$T_{J_2} + T_{K_1} + T_{B_1} + T_{B_2} = 0$$

$$\Rightarrow J_2 \frac{d^2\theta_2}{dt^2} + K_1(\theta_2 - \theta_1) + B_1 \frac{d(\theta_2 - \theta_1)}{dt} + B_2 \frac{d(\theta_2 - \theta_3)}{dt} = 0 \quad \text{—— (2)}$$

At J_3 :-



$$T_{J_3} = J_3 \frac{d^2\theta_3}{dt^2}$$

$$T_{B_2} = B_2 \frac{d(\theta_3 - \theta_2)}{dt}$$

$$T_{K_3} = K_3 (\theta_3 - \cancel{\theta_2}).$$

By Newtons law,

$$T_{J_3} + T_{K_3} + T_{B_2} = 0$$

$$\Rightarrow J_3 \frac{d^2\theta_3}{dt^2} + K_3 \theta_3 + B_2 \frac{d(\theta_3 - \theta_2)}{dt} = 0 \quad \text{—— (3)}$$

Now, the Equations (1), (2), (3) are the differential equations which are in angular displacements.

$\Rightarrow$ Now, $\frac{d\theta}{dt} = \omega$ (angular velocity)

$$\Rightarrow \theta = \int \omega \, dt$$

$$\frac{d^2\theta}{dt^2} = \frac{d\omega}{dt}$$

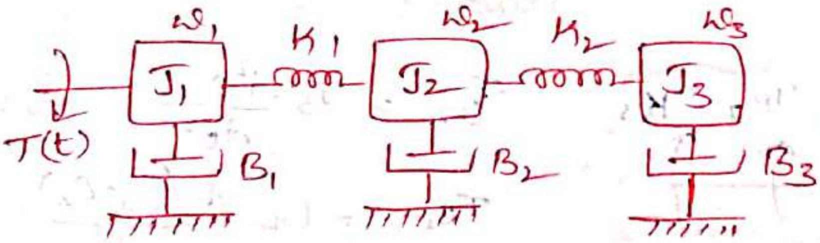
① ⇒ $J_1 \frac{d\omega_1}{dt} + K_1 \int (\omega_1 - \omega_2) dt + B_1(\omega_1 - \omega_2) = T(t)$. —— ④

② ⇒ $J_2 \frac{d\omega_2}{dt} + B_2(\omega_2 - \omega_3) + B_1(\omega_2 - \omega_1) + K_1 \int (\omega_2 - \omega_1) dt = 0$ —— ⑤

③ ⇒ $J_3 \frac{d\omega_3}{dt} + B_2(\omega_3 - \omega_2) + K_3 \int \omega_3 dt = 0$ —— ⑥

∴ Now, ④, ⑤, ⑥ are differential equations governing the given system, when angular velocities are given.

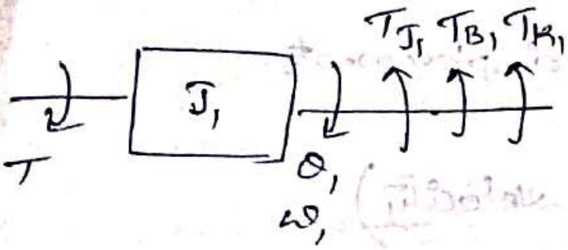
③ Write the Differential equations Regarding to the given System



Sol Since, from the given Rotational system, it has 3 inertia elements. So, 3 differential equations will be calculated. And given is angular velocities.

for J_1 :-

free body diagram will be,



$T_{J_1} = J_1 \frac{d\omega_1}{dt}$

$T_{B_1} = B_1 \cdot \omega_1$

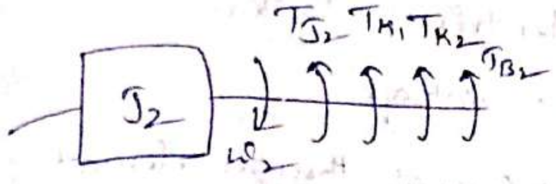
$T_{K_1} = K_1 \int (\omega_1 - \omega_2) dt$

By Newtons law,

⇒ $J_1 \frac{d\omega_1}{dt} + B_1 \cdot \omega_1 + K_1 \int (\omega_1 - \omega_2) dt = T(t)$ —— ①

for J_2 :-

free body diagram will be,



$\therefore T_{J_2} = J_2 \frac{d\omega_2}{dt}$; $T_{B_2} = B_2 \omega_2$

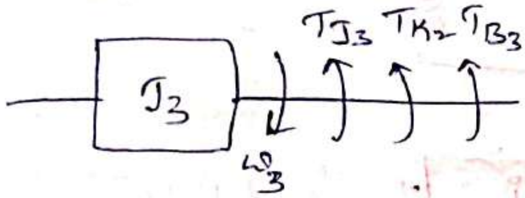
$T_{K_2} = K_2 \int (\omega_2 - \omega_3)\, dt$

$T_{K_1} = K_1 \int (\omega_2 - \omega_1)\, dt$

$\therefore$ Now, By Newtons law,

$$\Rightarrow J_2 \frac{d\omega_2}{dt} + B_2 \omega_2 + K_2 \int (\omega_2 - \omega_3)\, dt + K_1 \int (\omega_2 - \omega_1)\, dt = 0 \quad \text{— (2)}$$

for J_3 :-



$\therefore T_{J_3} = J_3 \frac{d\omega_3}{dt}$: $T_{B_3} = B_3 \omega_3$

$T_{K_2} = K_2 \int (\omega_3 - \omega_2)\, dt$

By Newtons law,

$$J_3 \frac{d\omega_3}{dt} + K_2 \int (\omega_3 - \omega_2)\, dt + B_3 \omega_3 = 0 \quad \text{— (3)}$$

$\therefore$ The equations (1), (2), (3) are the Differential Equations governing the given Mechanical Rotational System.

*Electrical Analogous of Rotational System :-

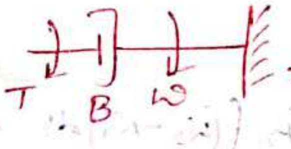
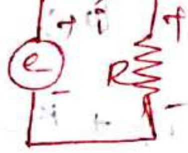
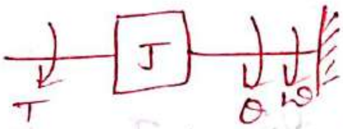
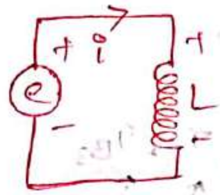
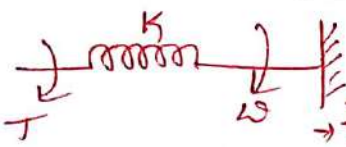
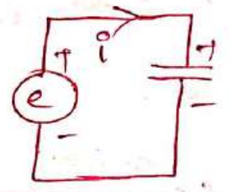
The 3 basic elements inertia, Dashpot and Spring are analogous to Resistance, inductance and Capacitance.

The input torque is analogous to Voltage & Current,

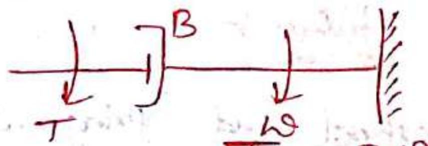
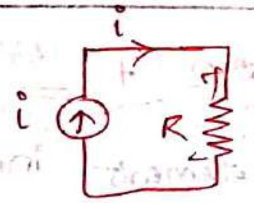
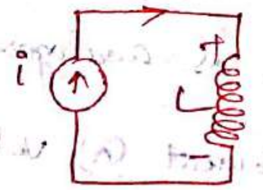
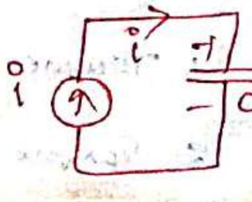
The output displacement (or) velocity is analogous to Voltage & Current Since, we have two Sources. So, we have two analogies ⇒ 1. Torque - Voltage analogy.

2. Torque - Current analogy.

* Torque - Voltage Analogy :-

Mechanical System	Electrical System
input :- applied Torque	input :- Voltage
output :- angular Velocity	output :- Current through element.
 $\Rightarrow T_B = B \cdot \omega$	 $\Rightarrow V = iR$
 $\Rightarrow T_J = J \frac{d\omega}{dt}$	 $\Rightarrow V = L \frac{di}{dt}$
 $\rightarrow T_K = K \int \omega dt$	 $\Rightarrow V = \frac{1}{C} \int i dt$

* Torque - Current analogy :-

Mechanical System.	Electrical System
input :- Torque	input :- Current
output :- angular Velocity.	output :- Voltage across element
 $T_B = B \cdot \omega$	 $\Rightarrow i = \frac{1}{R} V$
 $T_K = K \int \omega dt$	 $\Rightarrow i = \frac{1}{L} \int i dt$
 $\Rightarrow T_J = J \frac{d\omega}{dt}$	 $\Rightarrow i = C \frac{di}{dt}$

for force - Voltage analogy :-

Representation

$T(t) \longrightarrow V(t)$

$B \longrightarrow R$

$J \longrightarrow L$

$K \longrightarrow C$

Notation

$T(t) \longrightarrow V(t)$

$B \longrightarrow R$

$J \longrightarrow L$

$K \longrightarrow 1/C$

* for force - Current Analogy :-

Representation

$T(t) \longrightarrow i(t)$

$B \longrightarrow R$

$K \longrightarrow L$

$J \longrightarrow C$

Notation

$T(t) \longrightarrow i(t)$

$B \longrightarrow 1/R$

$K \longrightarrow 1/L$

$J \longrightarrow C$

* PROBLEMS :-

(1)

7/8/1

Unit - II

→ Transfer function of DC ServoMotor – AC Servomotor, – Synchro Transmitter and Reciever – Block diagram - Algebra – Signal -flow graph Reduction using MASON's Gain formula.

ServoMotors:

The Motors that are used in Automatic control systems are called ServoMotors. When the objective of the system is to control the position of the object then the system is called ServoMechanism. The ServoMotors are used to convert the electrical signal to the angular displacement of the shaft. There are variety of Servomotors available for control system Applications, The Suitability of motor for a particular application depends on the characteristics of the System, the purpose of the system and its operating conditions.

In general the main features of servomotors are,

1. Steady state stability.
2. wide range of speed control.
3. Low Mechanical and Electrical Inertia.
4. Fast Response.

Depending on the Supply Required to run the Motor, they are classified as (i) DC Servomotors and (ii) AC Servomotors.

The DC Servomotors are expensive than the AC Servomotors. But, the DC Servomotors have linear characteristics and so easy to Control.

The advantages of DC Servomotors are following.

1. Higher output than from a 50Hz motor of same size
2. Linearity of characteristics are achieved easily.
3. Easier speed control from zero speed to full speed in both directions.
4. Low electrical time constants (0.1 to 6ms) and low Mechanical time constants (2.3 to 40ms).

Applications:

The DC Servomotors are generally used for large power Applications such as Robotics and Machine tools.

AC Servomotors.

Advantages of AC Motors are lower cost, higher efficiency and less Maintainance, since there is no Commutator & brushes.

Applications.

∴ The AC Motors are best suited for low power Applications.

* DC Servomotors.

These are broadly classified as,

DC Servomotors.

- Sliding Contact Motors with Commutator & brushes
  - Permanent Magnet Motors
    1. Cylindrical armature motors.
    2. Disc armature Motors.
    3. Moving coil motors.
  - Electromagnetic field Motors.
    1. Armature Controlled Motors.
    2. Field Controlled Motors.
    3. Series Motor.
- Brushless (or) Contactless motors with transistor Commutator.

*Transfer function of Armature Controlled DC Motor :-

It is a DC Shunt Motor designed to satisfy the Requirement of DC Servomotor.

The Speed of DC Motor is directly proportional to armature voltage and inversly proportional to flux in field winding. The field is excited by a Constant DC Supply. This Speed Control System is an electro-Mechanical Control System. The electrical system Consists of Armature and Field circuit but only the armature circuit is Considered for analysis purpose, because the field is excited by Constant voltage.

This Can be Shown in following figure,

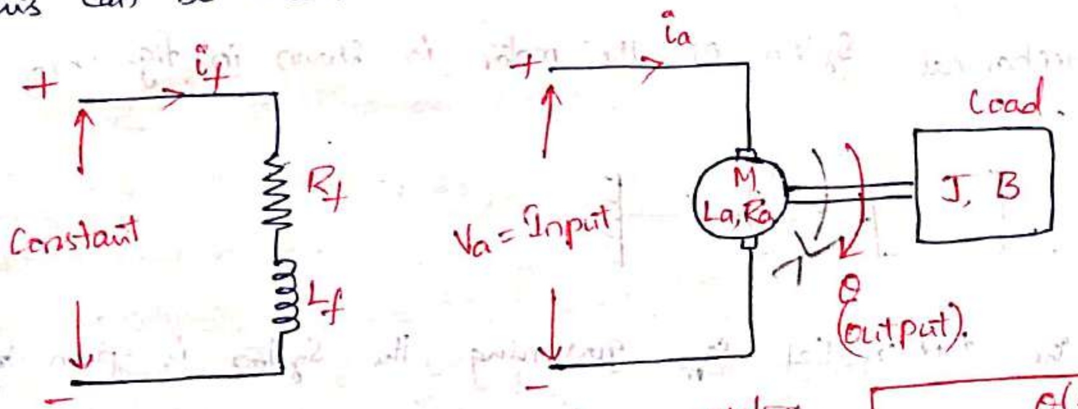


fig. Armature Controlled DC Motor

$$\boxed{T.F = \frac{\theta(s)}{V_a(s)}}$$

Some Approximations:-

let, R_a = Armature Resistance Ω.

L_a = Armature Inductance, H

i_a = Armature Current, A

V_a = Armature Voltage, V.

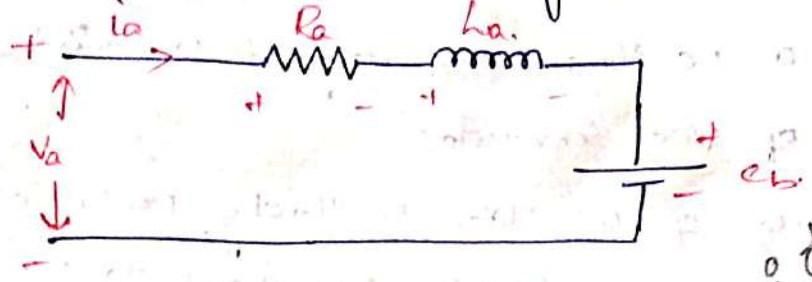
T = Torque Developed by Motor N-mts

θ = Angular Displacement of shaft, rad.

J = Movement of Inertia, Kg-m²/rad

B = Frictional Coefficient of Motor N-m/(rad/sec)

The equivalent circuit of above figure is given as,



By Kirchoff's voltage law,

$$i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \quad \text{--- (1)}$$

$T \propto \phi$

$T \propto i_a$

$T = K i_a$

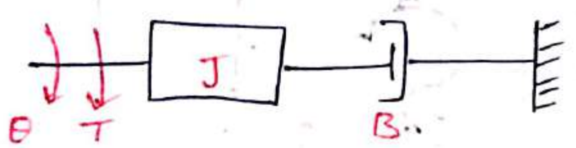
Torque of DC Motor is proportional to armature current.

$\Rightarrow T \propto i_a$ $\qquad T = 0.159 \phi I_a \frac{PZ}{A}$

$\Rightarrow$ Torque, $T = K_t i_a$ --- (2)

where, K_t = Torque Constant N-m/A

The Mechanical System of the motor is shown in fig. below,



Now, the Differential Eq. governing the System is given by,

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T \quad \text{--- (3)}$$

The back emf of DC Machine is proportional to Speed (angular velocity) of shaft.

$$\therefore e_b \propto \frac{d\theta}{dt} \Rightarrow e_b = k_b \frac{d\theta}{dt} \quad \text{--- (4)}$$

where, k_b = Back emf Constant

Now, the laplace transform of various time domain signals involved in this System,

$$L[V_a] = V_a(s), \quad L(e_b) = E_b(s), \quad L(T) = T(s).$$

$$L(i_a) = I_a(s), \quad L(\theta) = \theta(s).$$

Now applying the laplace transform for the above equations ①, ②, ③ and ④ we get,

① $\Rightarrow I_a(S) R_a + L_a S I_a(S) + E_b(S) = V_a(S)$. ——⑤

② $\Rightarrow K_t I_a(S) = T(S)$. ——⑥

③ $\Rightarrow J s^2 \theta(S) + BS\, \theta(S) = T(S)$. ——⑦

④ $\Rightarrow E_b(S) = K_b\, S\, \theta(S)$. ——⑧

Now, equating eq. ② and ③ i.e, ⑥ & ⑦ (Under Equilibrium Condition ⑥ = ⑦)

$$K_t I_a(S) = J s^2 \theta(S) + BS\, \theta(S)$$

$$\Rightarrow K_t I_a(S) = (J s^2 + BS)\, \theta(S).$$

$$\Rightarrow I_a(S) = \left[\frac{Js^2 + BS}{K_t}\right] \theta(S). \quad \text{——⑨}$$

The equation ⑤ can be written as,

⑤ $\Rightarrow (R_a + sL_a) I_a(S) + E_b(S) = V_a(S)$. ——⑩

then, Substituting the eq. ⑧ and ⑨ respectively in eq. ⑩

$$(R_a + sL_a)\left[\frac{Js^2 + BS}{K_t}\right]\theta(S) + K_b\, s\, \theta(S) = V_a(S).$$

$$\left[\frac{(R_a + sL_a)(Js^2 + BS) + K_b s \cdot K_t}{K_t}\right]\theta(S) = V_a(S).$$

Our Requirement, $T.F = \dfrac{\theta(S)}{V_a(S)}$.

$$\Rightarrow \boxed{T.F = \frac{\theta(S)}{V_a(S)} = \frac{K_t}{(R_a + sL_a)(Js^2 + BS) + (K_b \cdot K_t \cdot S)}} \quad \text{——⑪}$$

$$\frac{\theta(S)}{V_a(S)} = \frac{K_t}{R_a\left[\frac{sL_a}{R_a} + 1\right] BS\left(1 + \frac{Js^2}{BS}\right) + K_b K_t S.}$$

$\rightarrow$ T.F $= \dfrac{\theta(s)}{V_a(s)} = \dfrac{K_t / R_a B}{s\left[(1+sT_a)(1+sT_m) + \dfrac{K_b K_t}{R_a B}\right]}$

where $L_a/R_a = T_a$ = Electrical Time Constant

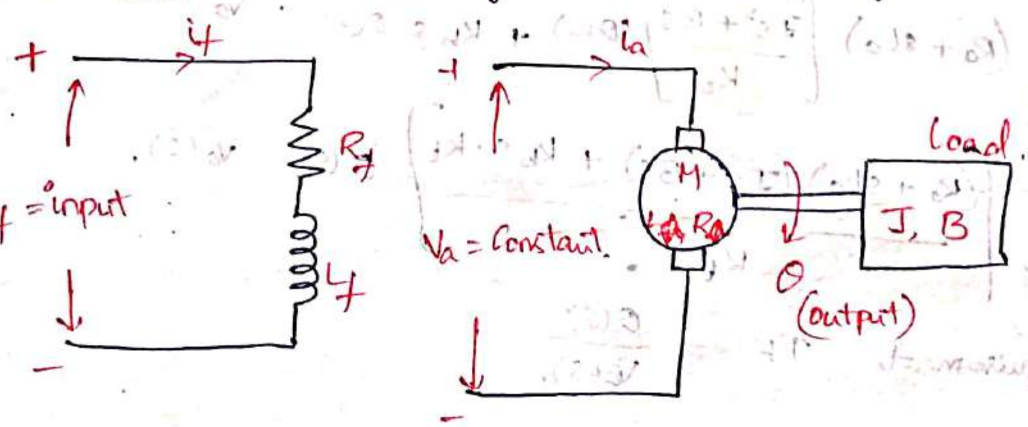
$J/B = T_m$ = Mechanical Time Constant

* Transfer function of Field Controlled DC Motor :-

It is a DC Shunt motor designed to satisfy the Requirement of the DC Servomotor. The Speed of DC Motor is directly Proportional to armature voltage and inversely proportional to flux.

- In field Controlled DC Motor the armature voltage is kept Constant and the Speed is varied by varying the flux.

The Speed Control System is an electro Mechanical Control System. The electrical System Consist of armature and field circuit but for the analysis purpose the field circuit to be considered since, the armature is excited by a Constant voltage



let R_f = Field Resistance, Ω

L_f = Field inductance, H

i_f = Field Current, A

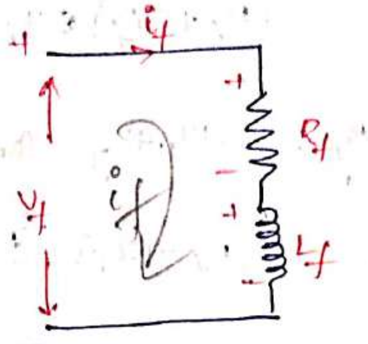
V_f = Field Voltage, V

T = Torque developed by the Motor N-m.

J = Movement of inertia of Motor kg-m²/rad.

B = Frictional Coefficient of Motor N-m/(rad/sec).

The equivalent ckt of field is given,



The Kirchoff's law,

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad \text{——— (1)}$$

Torque Developed by Motor is proportional to the field Current.

Equivalent ckt of field.

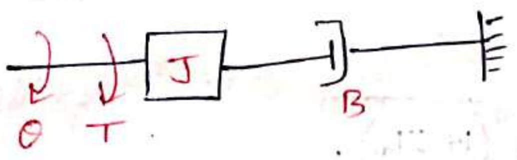
$\Rightarrow T \propto i_f \quad \Rightarrow T = K_{tf}\, i_f$ ——— (2)

$T \propto i_a \phi$

$T \propto \phi\, i_f$

where, K_{tf} = Torque Constant N-m/A.

equivalent circuit of the Mechanical System is given below,



The differential Equation governing to this system is given by,

$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T. \quad \text{——— (3)}$$

Taking the laplace transform of Various time domain signals,

$L(i_f) = I_f(s)$, $L(T) = T(s)$, $L(V_f) = V_f(s)$, $L(\theta) = \theta(s)$.

Now, the equations of (1), (2), (3) becomes.

(1) $\Rightarrow R_f I_f(s) + L_f s \cdot I_f(s) = V_f(s).$ ——— (4)

(2) $\Rightarrow T(s) = K_{tf}\, I_f(s).$ ——— (5)

(3) $\Rightarrow J s^2 \theta(s) + Bs\, \theta(s) = T(s)$ ——— (6)

Now equating the eq. (5) and (6) will Results to,

$$K_{tf}\, I_f(s) = J s^2 \theta(s) + Bs\, \theta(s).$$

$$\Rightarrow I_f(s) = \frac{(Js^2 + Bs)\, \theta(s)}{K_{tf}} \quad \text{——— (7)}$$

The equation (4) can be written as,

$$(R_f + sL_f)\, I_f(s) = V_f(s). \quad \text{—— (8)}$$

By substituting the eq. (7) in (8) we get

$$(R_f + sL_f)\,\frac{(Js+B)s}{K_{tf}}\,\theta(s) = V_f(s).$$

$$\Rightarrow \therefore \text{The } \boxed{T.F = \frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{s(R_f+sL_f)(B+sJ)}}.$$

$$\Rightarrow \frac{\theta(s)}{V_f(s)} = \frac{K_{tf}}{sR_f\left(1+\frac{sL_f}{R_f}\right)B\left(1+\frac{sJ}{B}\right)}$$

$$= \frac{K_{tf}/R_f B}{s(1+sT_f)(1+sT_m)}.$$

where, $T_f = L_f/R_f$ = Field time constant

$T_m = J/B$ = Mechanical time constant

* Construction of Block Diagram from the Transfer function of Armature Controlled DC Motor

The Differential equations governing the Armature Controlled DC Motor are

$$V_a = i_aR_a + L_a\frac{di_a}{dt} + e_b. \quad \text{—— (1)}$$

$$T = K_t\, i_a. \quad \text{—— (2)}$$

$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} \quad \text{—— (3)}$$

$$e_b = K_b\frac{d\theta}{dt}. \quad \text{—— (4)}$$

Now, assuming the term $\frac{d\theta}{dt} = \omega$. —— (A) then the above equations will Results to,

$$V_a = i_a R_a + L_a \frac{di_a}{dt} + e_b. \quad (5)$$

$$T = K_t i_a. \quad (6)$$

$$T = J \cdot \frac{d\omega}{dt} + B\omega. \quad (7)$$

$$e_b = K_b \cdot \omega \quad (8)$$

Now taking Laplace transform for above equations.

$$\Rightarrow V_a(s) = I_a(s) R_a + L_a \cdot s \cdot I_a(s) + E_b(s). \quad (9)$$

$$\Rightarrow T(s) = K_t \cdot I_a(s). \quad (10)$$

$$\Rightarrow T(s) = Js\,\omega(s) + B\,\omega(s).$$

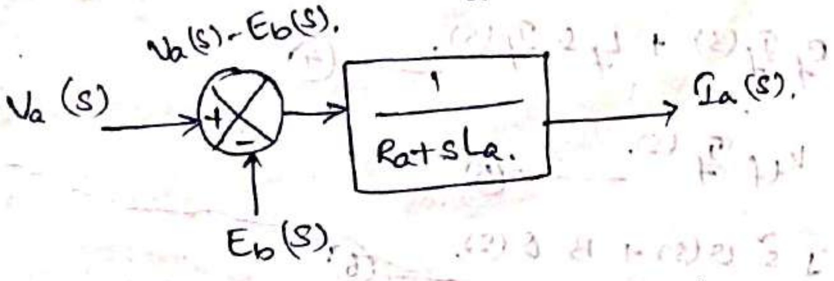
$$= (Js + B)\,\omega(s). \quad (11)$$

$$\Rightarrow E_b(s) = K_b \cdot \omega(s). \quad (12)$$

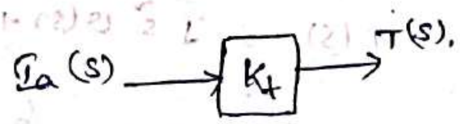
Now, (9) $\Rightarrow V_a(s) - E_b(s) = I_a(s) R_a + L_a \cdot s \cdot I_a(s).$

$$\Rightarrow I_a(s) = \frac{1}{R_a + s \cdot L_a} \left[V_a(s) - E_b(s) \right]$$

[∵ Taking the voltage terms to one side]

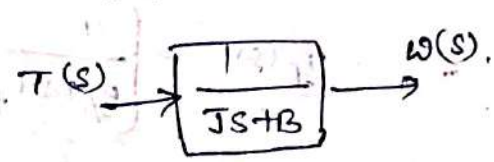


$\Rightarrow$ (10) $T(s) = K_t \cdot I_a(s).$ $\Rightarrow$

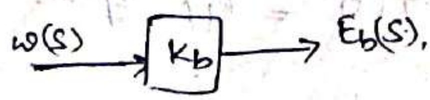


$\Rightarrow$ (11) $T(s) = (Js + B)\,\omega(s).$

$$\Rightarrow \omega(s) = \frac{1}{Js + B} T(s).$$



$\Rightarrow$ (12) $\Rightarrow E_b(s) = K_b \cdot \omega(s).$



and finally, we have assumed that from eq (A)

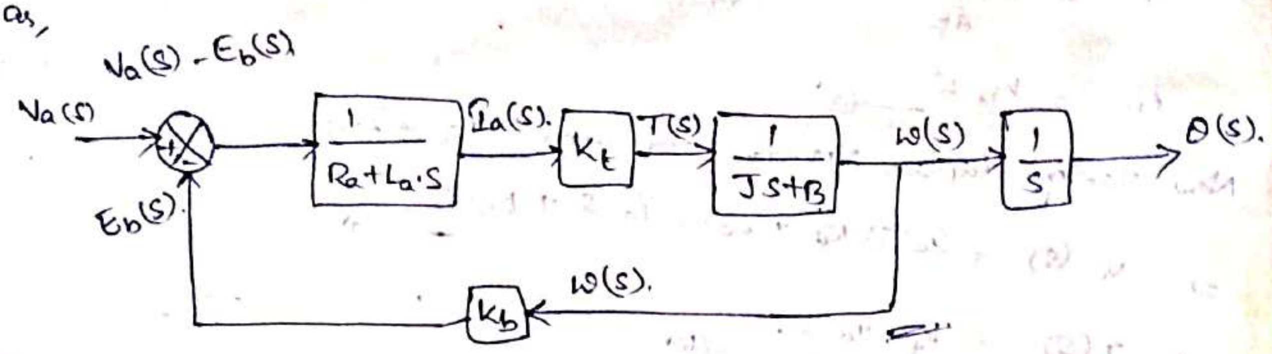
$$\Rightarrow (A) \Rightarrow \frac{d\theta}{dt} = \omega.$$

Laplace transform $\Rightarrow$ $s \cdot \theta(s) = \omega(s)$.

$\Rightarrow$ $\theta(s) = \frac{1}{s}\, \omega(s)$

$$\omega(s) \rightarrow \boxed{\frac{1}{s}} \rightarrow \theta(s).$$

Now Combining all the blocks we get, the block diagram as,



Similarly for Field Controlled DC Motor

the Differential Equations

$$V_f = R_f i_f + L_f \frac{di_f}{dt} \quad \text{—— ①}$$

$$T = K_{tf}\, i_f \quad \text{—— ②}$$

$$T = J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} \quad \text{—— ③}$$

Taking laplace transform for above equations

$$\Rightarrow V_f(s) = R_f I_f(s) + L_f s \cdot I_f(s). \quad \text{—— ④}$$

$$T(s) = K_{tf}\, I_f(s). \quad \text{—— ⑤}$$

$$T(s) = J s^2 \theta(s) + B\, \theta(s). \quad \text{—— ⑥}$$

④ $\Rightarrow$ $V_f(s) = I_f(s)\,(R_f + s \cdot L_f)$

$\Rightarrow$ $I_f(s) = \left[\frac{1}{R_f + s \cdot L_f}\right] V_f(s)$.

$$V_f(s) \rightarrow \boxed{\frac{1}{R_f + s \cdot L_f}} \rightarrow I_f(s).$$

⑤ $\Rightarrow$ $T(s) = K_{tf} \cdot I_f(s)$.

$$\Rightarrow I_f(s) \rightarrow \boxed{K_{tf}} \rightarrow T(s).$$

⑥ $\Rightarrow$ $T(s) = (Js^2 + Bs)\, \theta(s)$.

$\Rightarrow$ $\theta(s) = \left[\frac{1}{Js^2 + Bs}\right] T(s)$.

$$\Rightarrow T(s) \rightarrow \boxed{\frac{1}{Js^2 + Bs}} \rightarrow \theta(s).$$

Now, Combining all the blocks we get

$$V_f(s) \longrightarrow \boxed{\frac{1}{R_f + s\cdot L_f}} \xrightarrow{I_f(s)} \boxed{K_{tf}} \xrightarrow{T(s)} \boxed{\frac{1}{Js^2 + Bs}} \longrightarrow \theta(s)$$

NOTE:

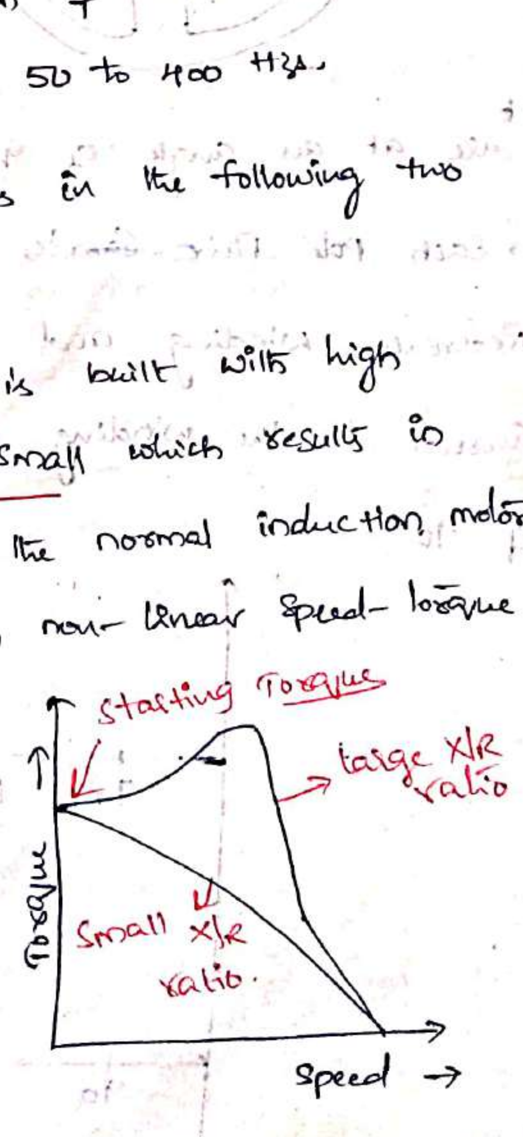
1. Comparison of AC & DC Servomotors and the Comparison of Field Controlled & Armature Controlled Motors are in V.U Bakshi & U.A. Bakshi page No:- 4.35

* AC Servomotors:

Most of the Servomotors used in low power Servo-Mechanisms are of A.C. Servomotors. An A.C. Servomotor is basically of 2-phase induction motor. The output power of A.C. Servomotor is varies from fraction of watt to few hundreds of watts. The operating frequency is 50 to 400 Hz.

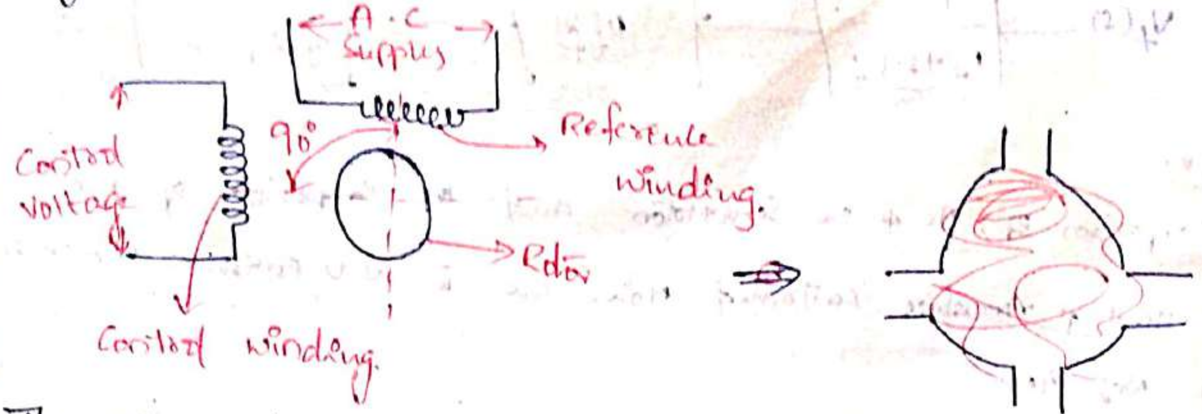
A two-phase servomotor differs in the following two ways from Normal Induction Motor

1. The Rotor of A.C. Servomotor is built with high Resistance, so the X/R ratio is small which results in mostly linear characteristics. But, the normal induction motors will have high X/R ratio results in non-linear speed-torque characteristics.

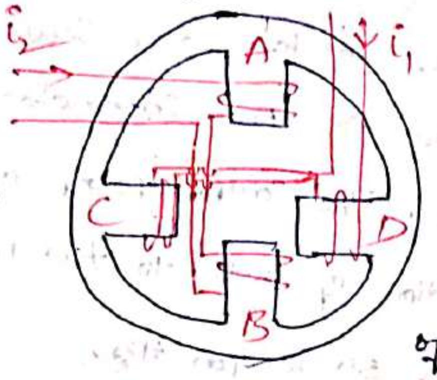


2. The Excitation voltage applied to 2-stator windings should have the phase difference of 90°.

It is mainly divided into stator & Rotor. The schematic diagram of stator is shown below.

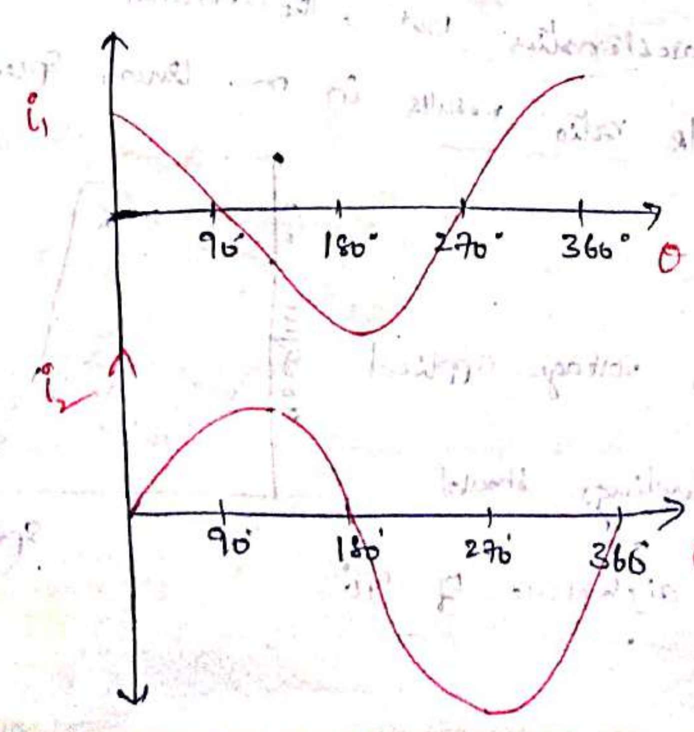


The above diagram can also be shown as,



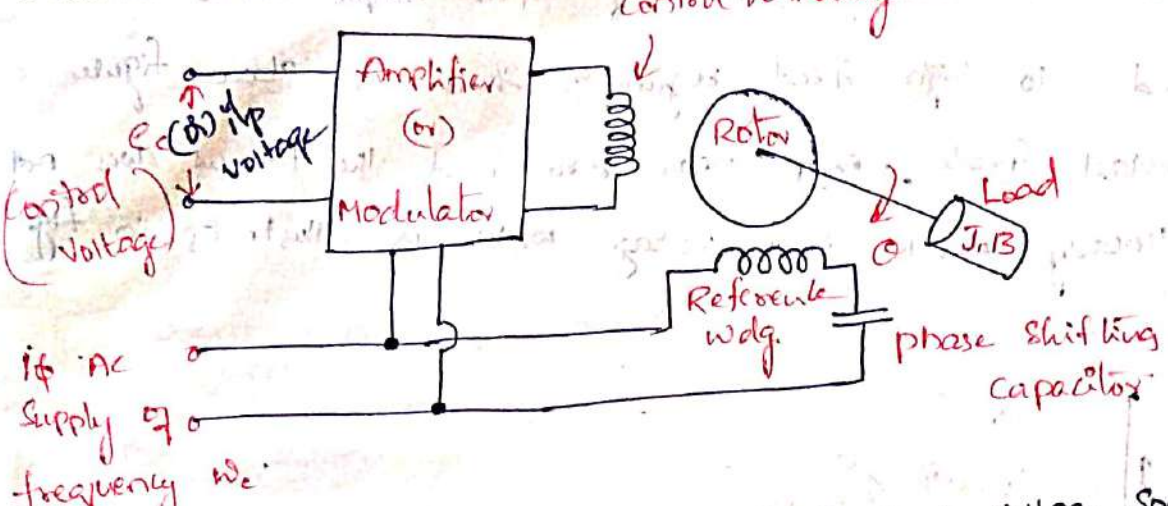
→ The stator consists of two pole pairs (A-B and C-D) mounted on the inner pheriphery of the stator, such that there axes are at an angle of 90° in space.

→ Each pole pair carries a winding. One winding is called Reference winding and other is control winding. The exciting current in the winding should have a phase displacement of 90°.



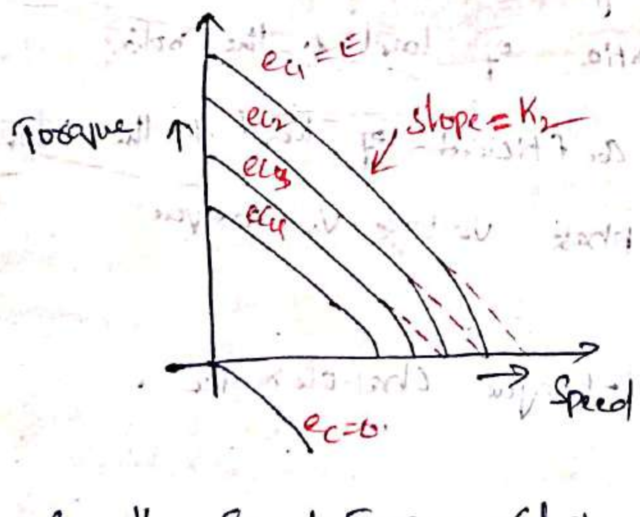
Exciting Currents

Now Symbolic Representation of an AC Servomotor is shown below.



The Reference wdg is excited by a Constant voltage Source with a frequency in the range 50 to 1000 Hzs. By using frequencies of 400 Hzs or higher, the system can be made less ~~harmful~~ influence to low-frequency noise. Due to this feature, ac devices are extensively used in aircraft and missile Control systems.

The Speed-Torque Curves of typical AC Servomotors are plotted below.



phase shifting capacitor is used to shift the excitation voltages to phase displacement of 90°.

So, the Speed-Torque Characteristics of AC Servomotor are non-linear except in low-speed region. In order to derive the transfer function for the motor, some linearizing approximations are necessary. AC Servomotor not frequently operates in high speed therefore, the linear portions of speed-torque curves can be extended to high speed region as shown in above figure. At constant speeds, except near zero speed, the torque does not vary linearly w.r.t input voltage which is illustrates in fig below.

Control-phase voltage (or) i/p voltage →

*** Transfer function of AC Servomotor :-**

Let, T_M = Torque developed by servomotor

θ = Angular displacement of rotor

$\omega = \frac{d\theta}{dt}$ = Angular speed.

T_L = Torque required by the load.

J = Movement of inertia of load & the rotor

B = Viscous-Frictional coefficient of load & the rotor

K_1 = Slope of control phase voltage Vs. Torque characteristic.

K_2 = slope of speed-torque characteristic.

With Reference to the above characteristic we can say that for speeds near zero, all the curves are straight lines parallel to the characteristic of rated input voltage $(e_c = E)$

and are Equally Spaced for equal increments of the input voltage. Under this assumption, the Torque developed by the motor is expressed as,

Torque Developed by Motor $T_m = K_1 e_c - K_2 \frac{d\theta}{dt}$

The Rotating part of motor and the load can be modelled as the equation

load Torque, $T_L = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt}$

At equilibrium the motor torque is equal to load torque.

$$\therefore J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_1 e_c - K_2 \frac{d\theta}{dt}$$

Now, by taking the laplace transform for the above equation we get

$$\Rightarrow JS^2\theta(S) + BS\theta(S) = K_1 E_c(S) - K_2 S\theta(S)$$

$$\Rightarrow \theta(S)(JS^2 + BS + K_2 S) = K_1 E_c(S).$$

$$\therefore \text{The } \boxed{T.F = \frac{\theta(S)}{E_c(S)} = \frac{K_1}{S(JS+B+K_2)}}$$

$$\Rightarrow \frac{\theta(S)}{E_c(S)} = \frac{K_1}{S\left(\frac{JS}{B+K_2}+1\right)\times(B+K_2)}$$

$$= \frac{K_1 / B+K_2}{S\left(\frac{J}{B+K_2}S+1\right)} = \frac{K_m}{S(1+T_m S)}$$

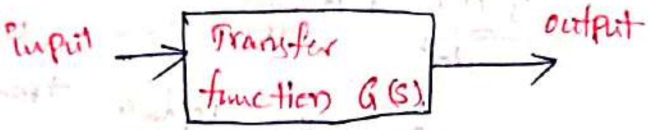
where, K_m = Motor gain Constant, T_m = motor Time Constant.

* Block Diagrams:-

A Control System may Consists of a number of Components. In Control Systems Engineering to show the functions performed by each Component, we Commonly use a diagram Called the Block diagram. The basic elements of a block diagram are 1. Block 2. Branch point and 3. Summing point.

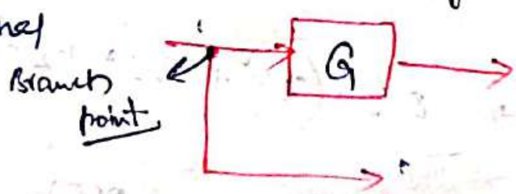
Block:-

In a block diagram all the System variables are linked to each other with the help of functional Blocks. The functional block (or) simply blocks is a symbol for the mathematical operation on the input signal to the block that produces the output



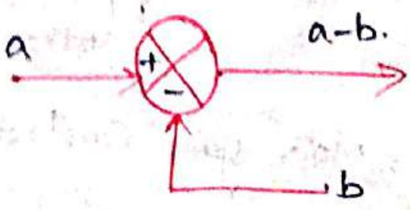
Branch point:-

A Branch point is a point from which the signal from a block goes Concurrently to other blocks or Summing points used to Carry a particular signal to any other blocks & to any Summing points



Summing point:-

Summing points are used to add two or more Signals in the System. The plus (or) minus sign at each arrowhead indicates whether the signal is to be added (or) to be Substracted



Constructing Block diagram for Control System:-

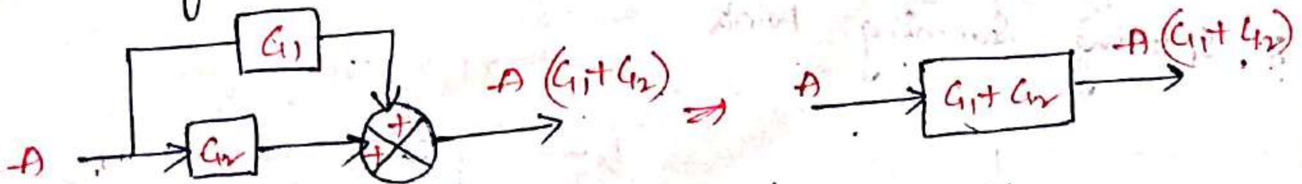
A Control system can be represented diagramatically by block diagram. The differential equations governing the system are used to construct the block diagram.

Each equation gives one section of block. The output of one section will be the input of another section. The various sections are interconnected to obtain the overall block diagram of the system.

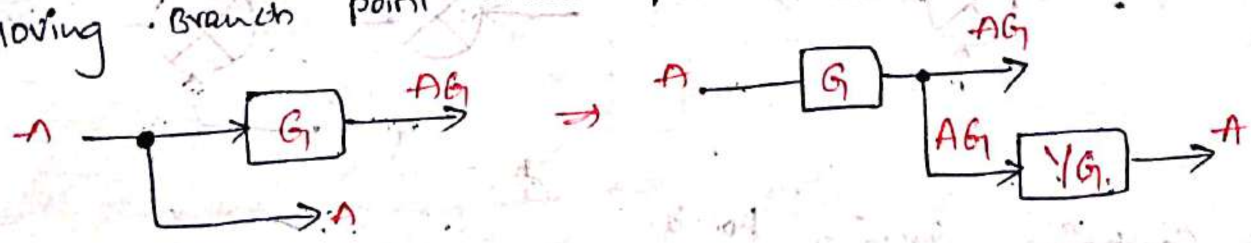
Rules of Block diagram Algebra :-

1. Combining the blocks in cascade

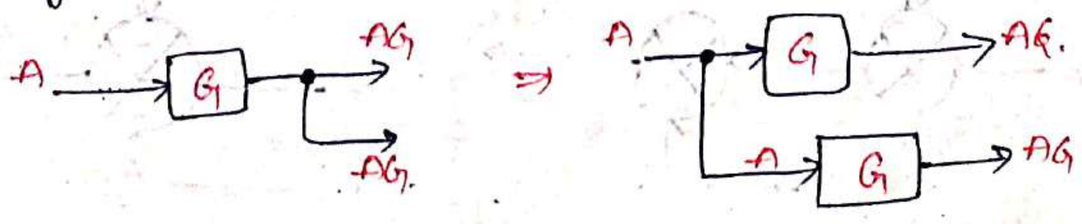
2. Combining the parallel Blocks:-



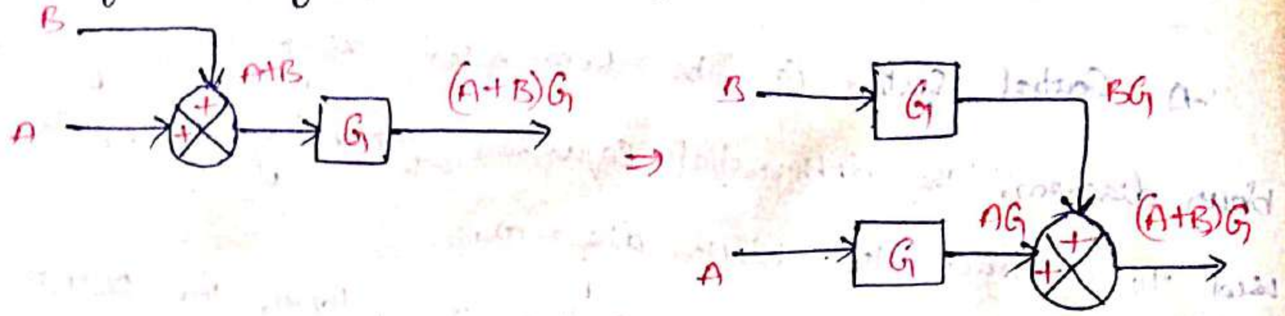
3. Moving Branch point ahead of block.



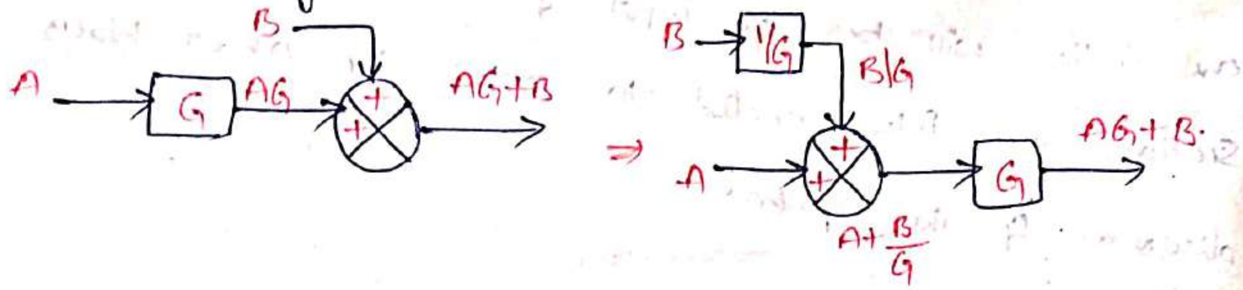
4. Moving the Branch point before the block.



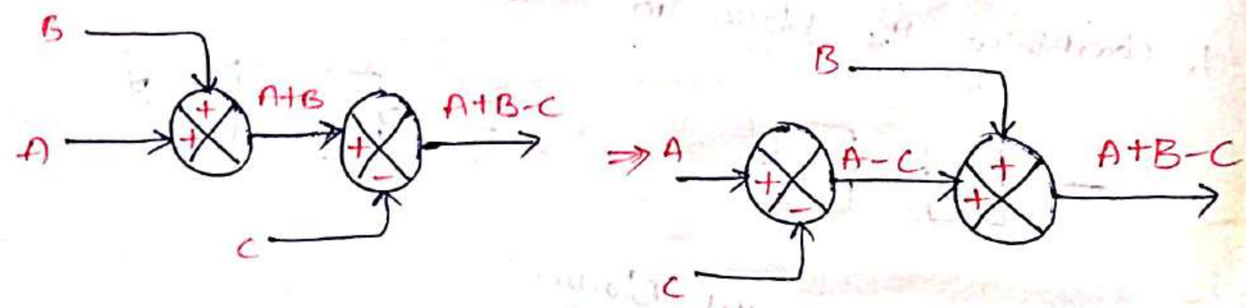
5. Moving Summing point ahead of the block.



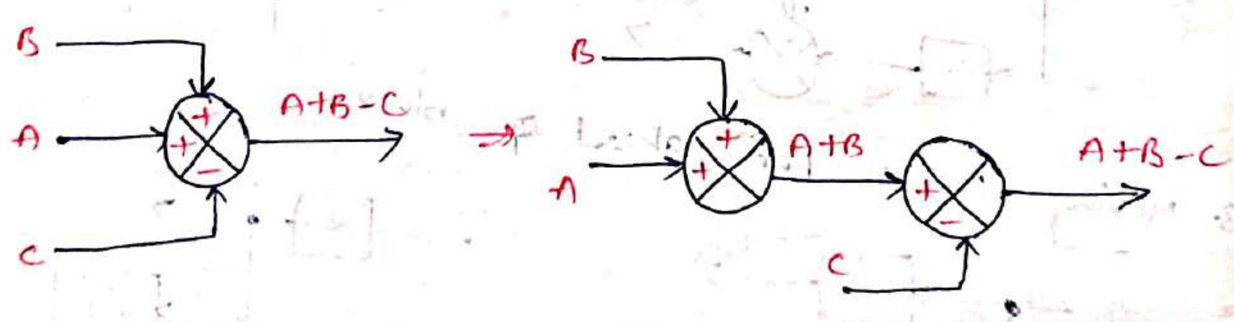
6. Moving Summing point before the block.



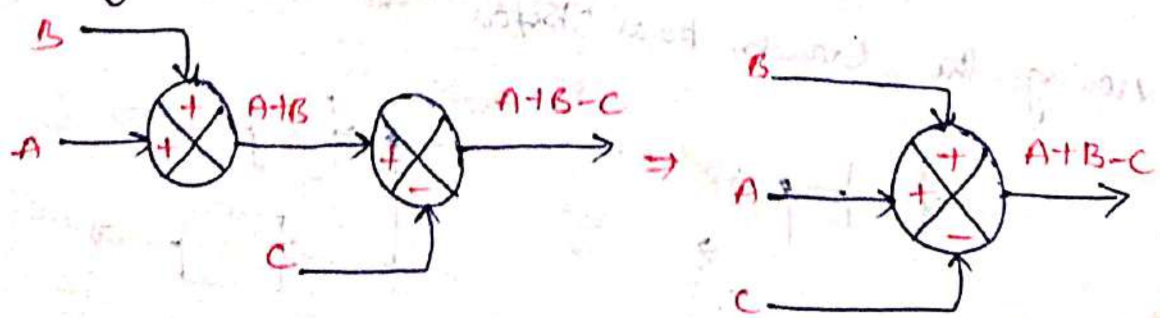
7. Interchanging Summing point



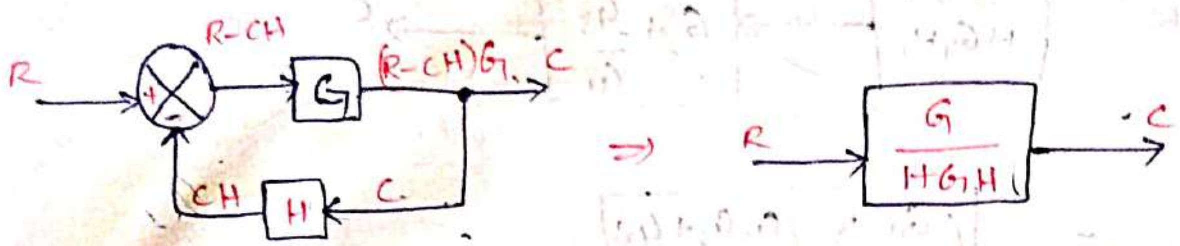
8. Splitting Summing points.



9. Combining Summing points.



10. Elimination of feedBack loop.



Proof: $C = (R-CH)G_1$

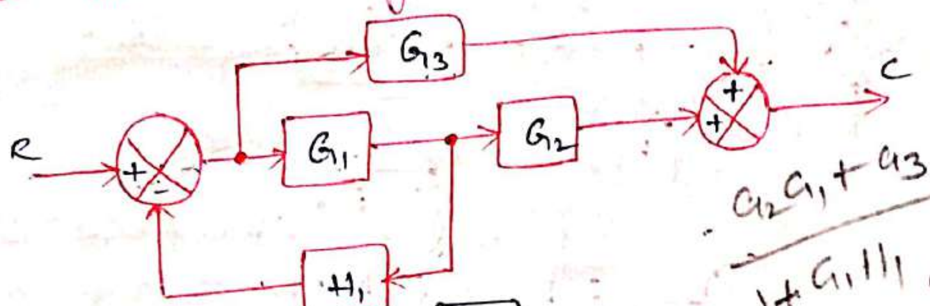
$C = RG_1 - CHG_1$

$\Rightarrow C(1+HG_1) = RG_1$

$$\Rightarrow \boxed{\frac{C}{R} = \frac{G_1}{1+HG_1}}$$

Problems:

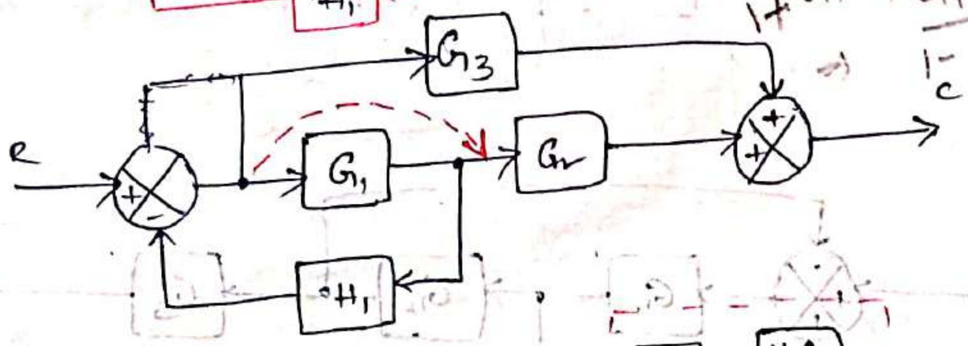
1. Reduce the block diagram to find C/R.

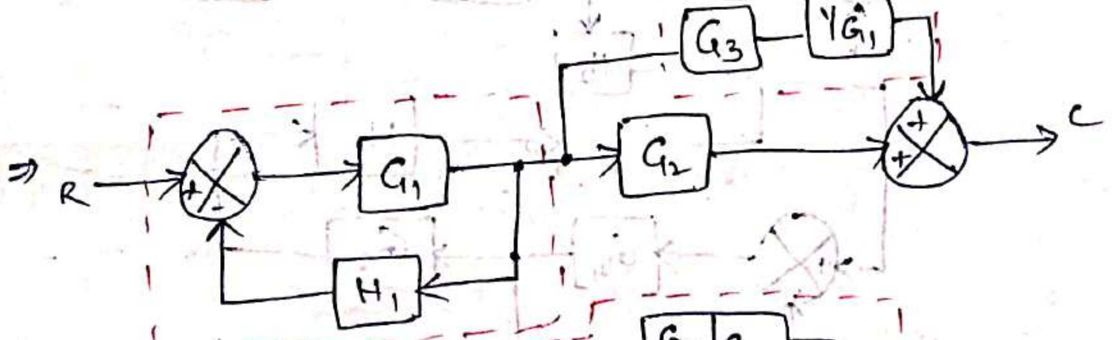


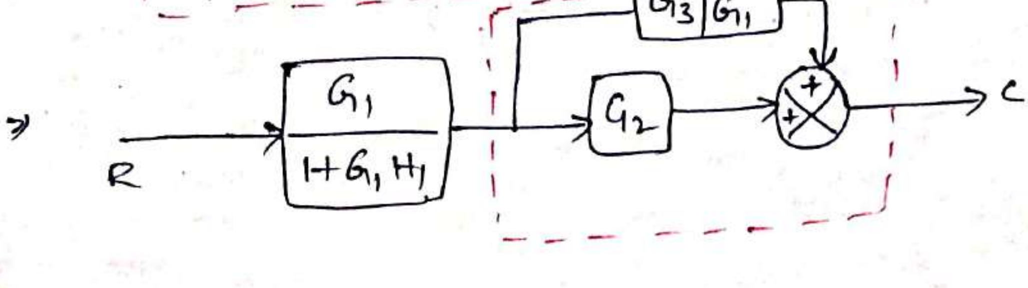
$$\frac{G_2G_1+G_3}{1+G_1H_1}$$

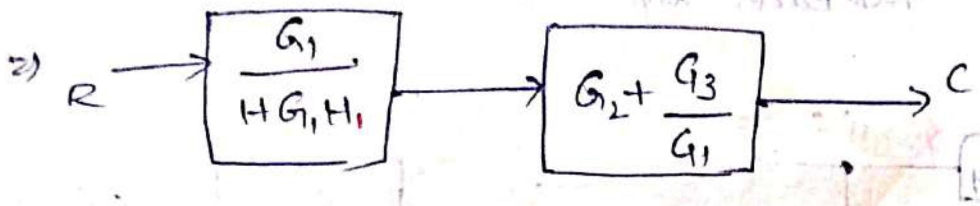
$$\frac{G_1G_2+G_3}{1-(-G_1H_1)}$$

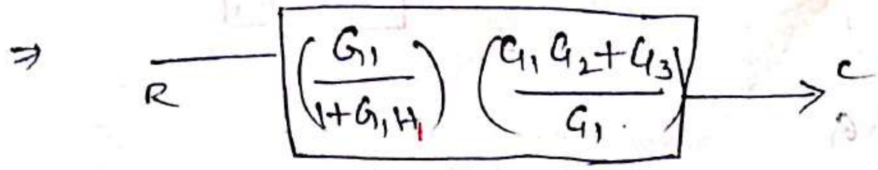
Sol:







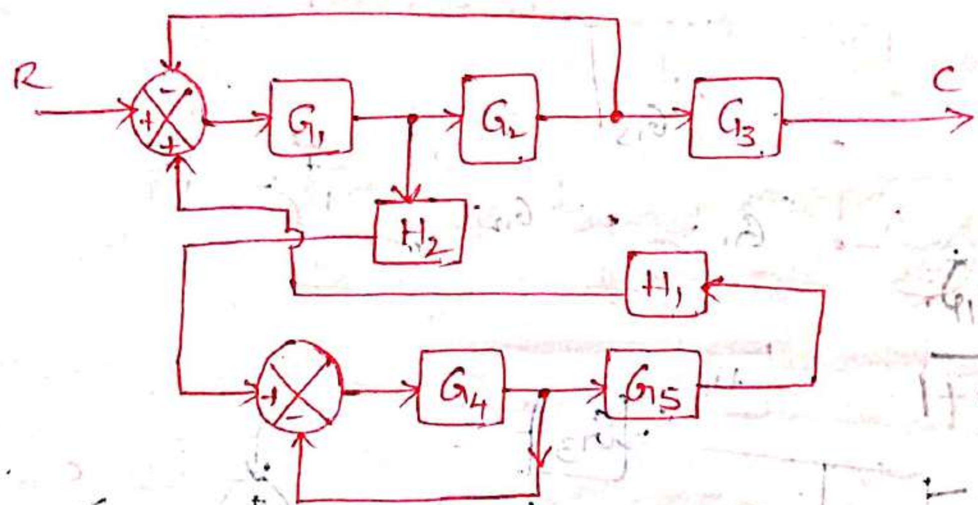




$$\Rightarrow T.F = \frac{C}{R} = \left(\frac{\cancel{G_1}}{1+H_1G_1}\right)\left(\frac{G_1G_2+G_3}{\cancel{G_1}}\right)$$

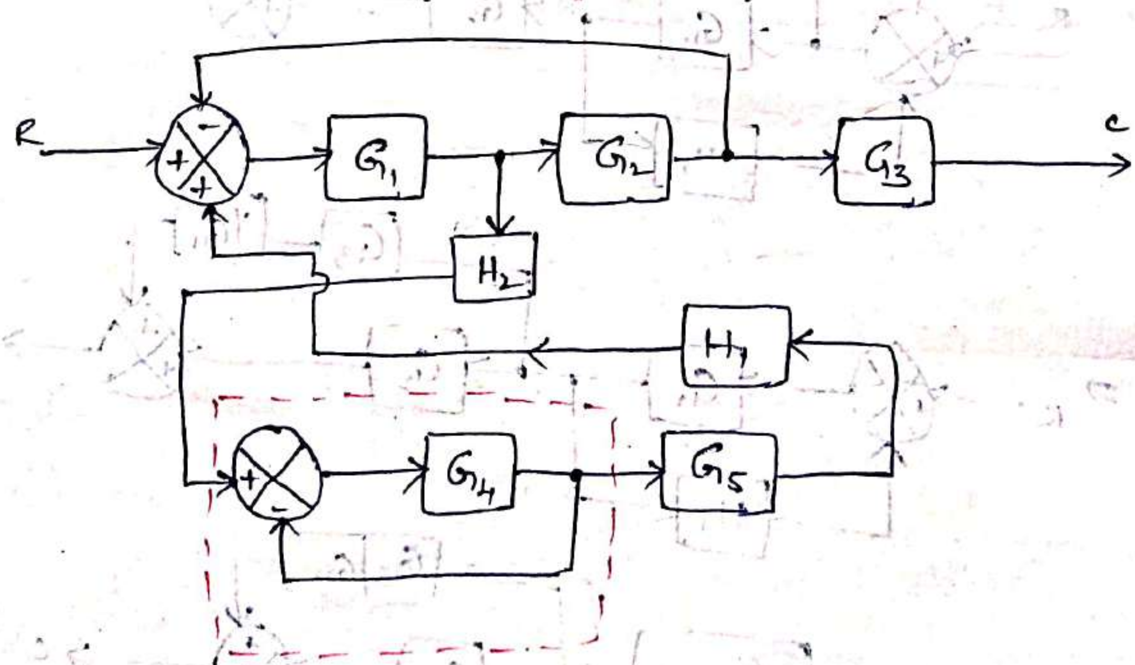
$$\therefore \boxed{\frac{C}{R} = \frac{G_1G_2+G_3}{1+G_1H_1}}$$

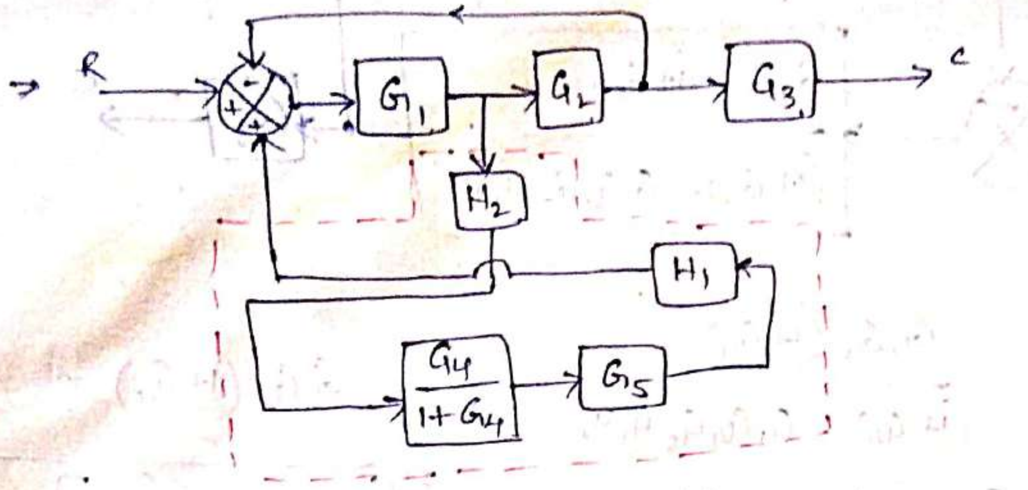
(2)

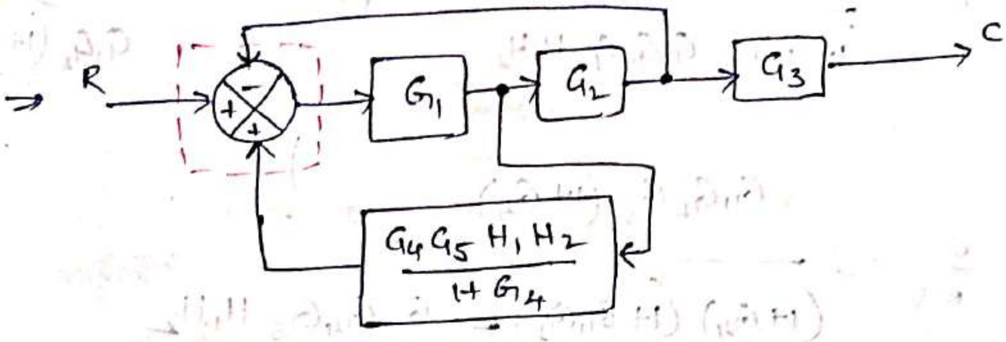


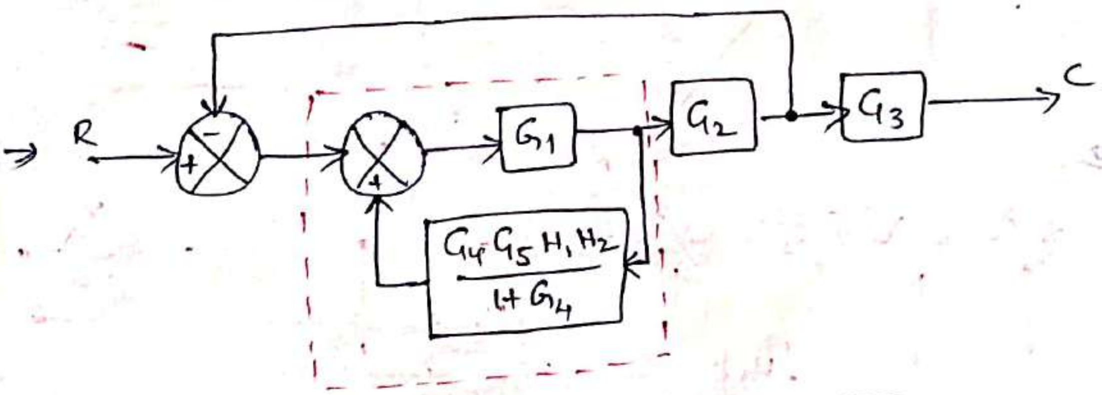
$$\frac{G_1G_2G_3}{1+G_1G_2+G_4 - G_4G_5H_1H_2}$$

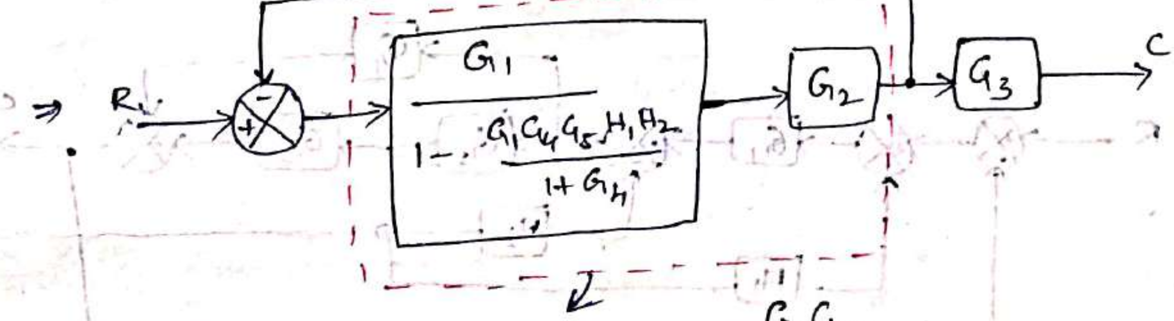
Sol







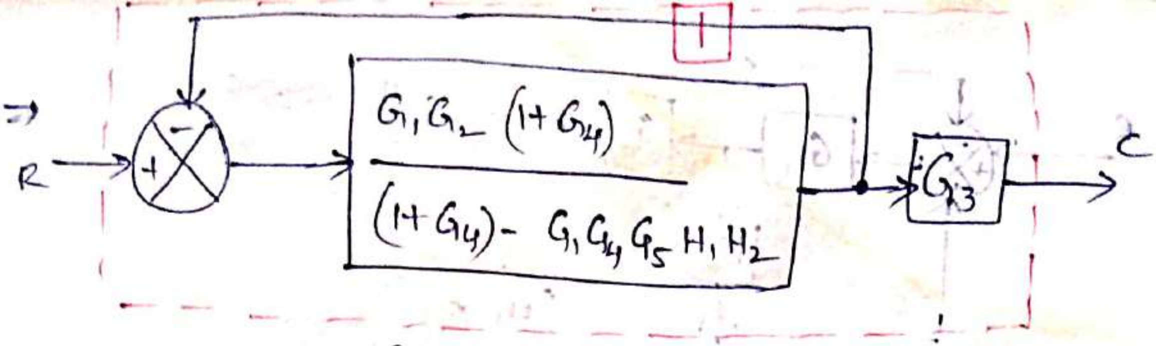




$$\Rightarrow \frac{G_1}{1-\frac{G_1G_4G_5H_1H_2}{1+G_4}} \times G_2 \Rightarrow \frac{G_1G_2}{1+G_4-\frac{G_1G_4G_5H_1H_2}{1+G_4}}$$

$$\Rightarrow \frac{G_1G_2\,(1+G_4)}{(1+G_4)-G_1G_4G_5H_1H_2}$$

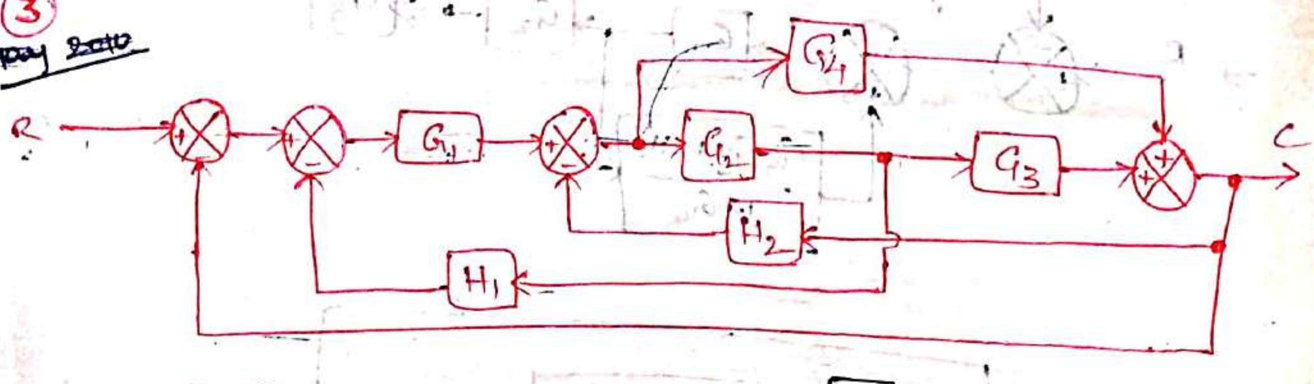
$\Rightarrow$



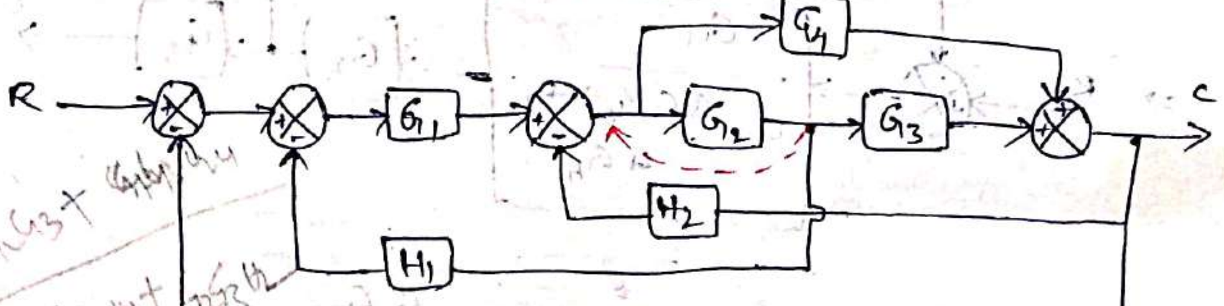
$$\Rightarrow \frac{\dfrac{G_1G_2(1+G_4)}{(1+G_4)-G_1G_4G_5H_1H_2}}{1+\dfrac{G_1G_2(1+G_4)}{(1+G_4)-G_1G_4G_5H_1H_2}} \Rightarrow \frac{G_1G_2(1+G_4)}{\left[(1+G_4)-G_1G_4G_5H_1H_2\right]+G_1G_2(1+G_4)}$$

$$\therefore \boxed{T.F = \frac{C}{R} = \frac{G_1G_2G_3(1+G_4)}{(1+G_4)(1+G_1G_2)-G_1G_4G_5H_1H_2}}$$

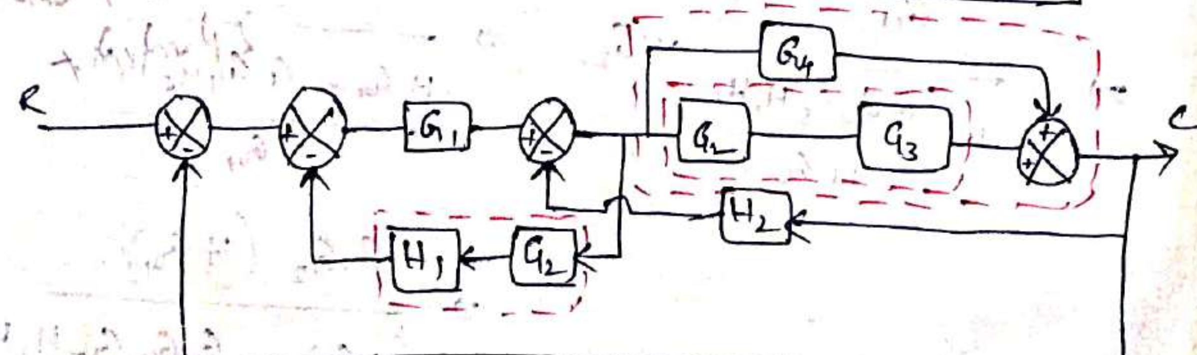
③ May 2010

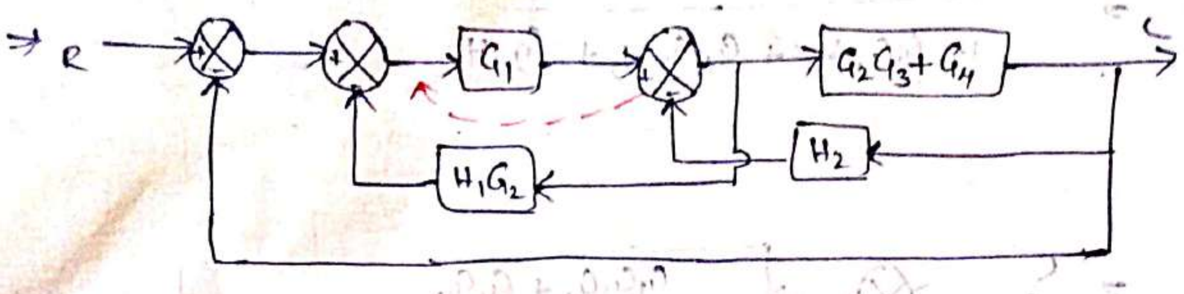


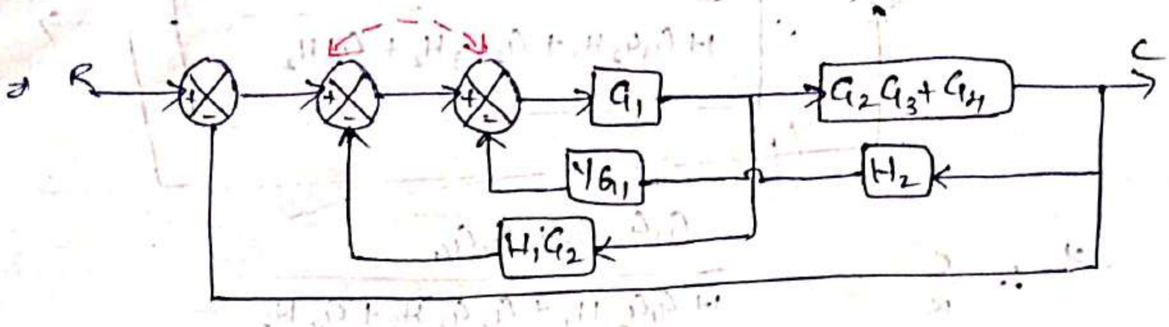
Sol.

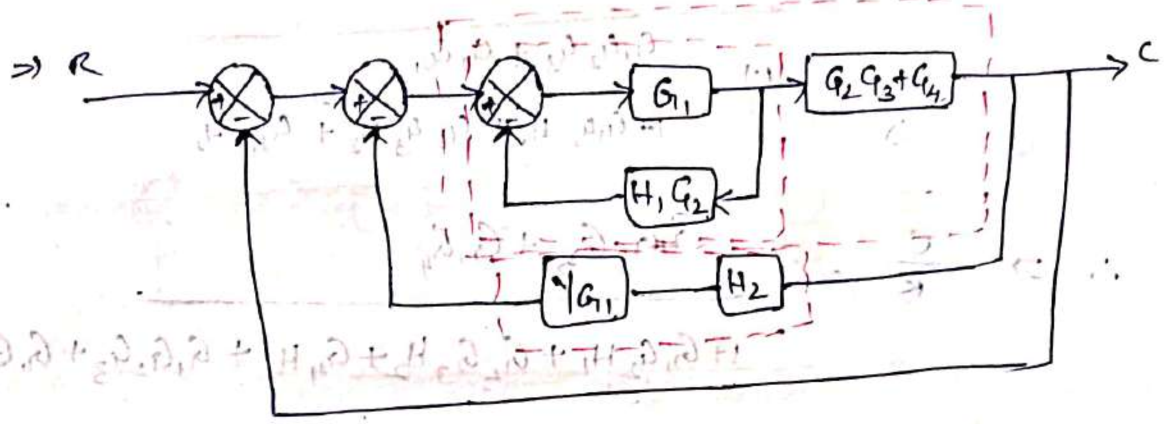


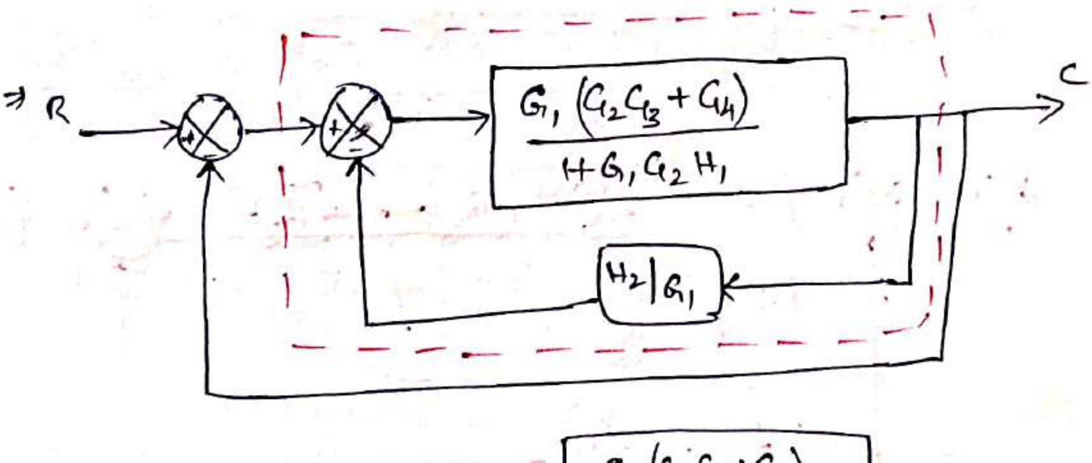
$\Rightarrow$

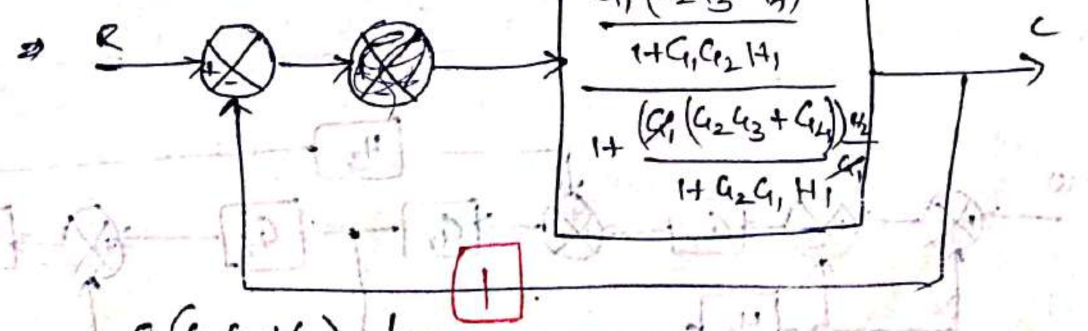






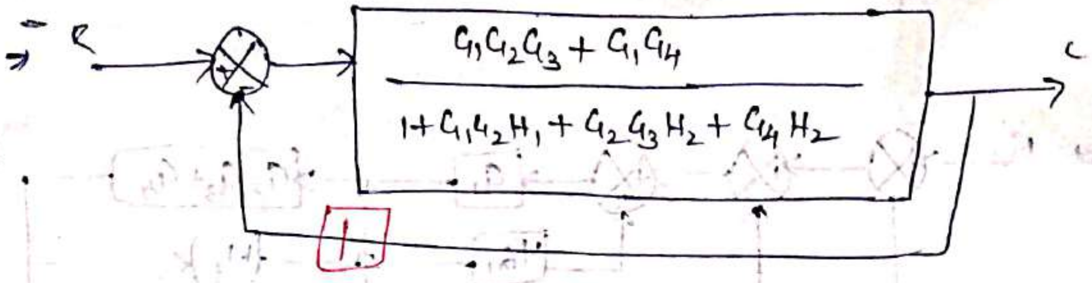






$$\Rightarrow \frac{G_1(G_2G_3+G_4)}{1+G_1G_2H_1} \Bigg/ \frac{(1+G_1G_2H_1)+(\cancel{G_1}G_2G_3+G_4)H_2}{1+G_2G_1H_1}$$

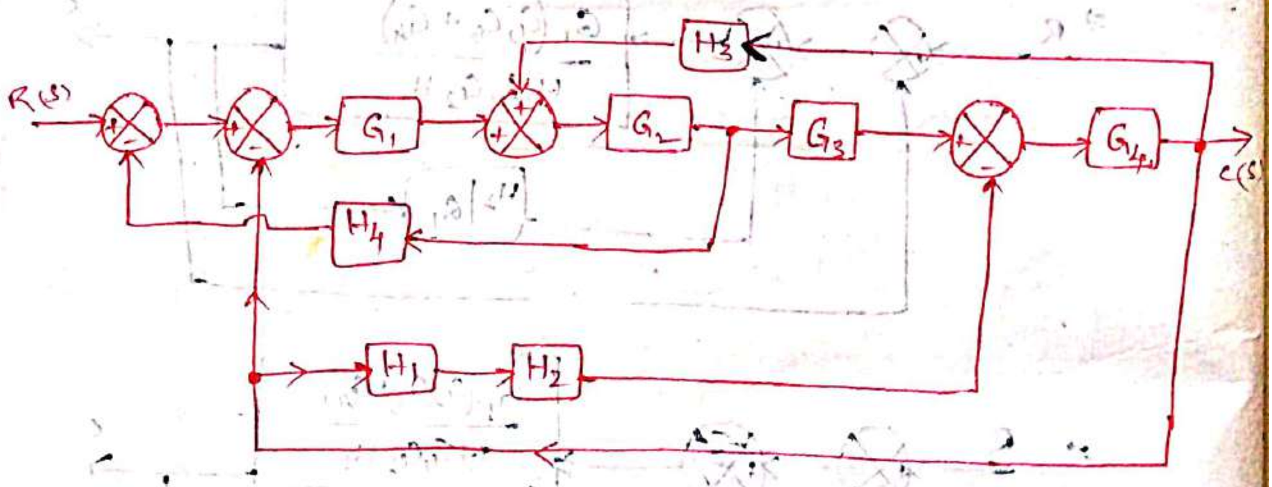
$$\Rightarrow \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}$$

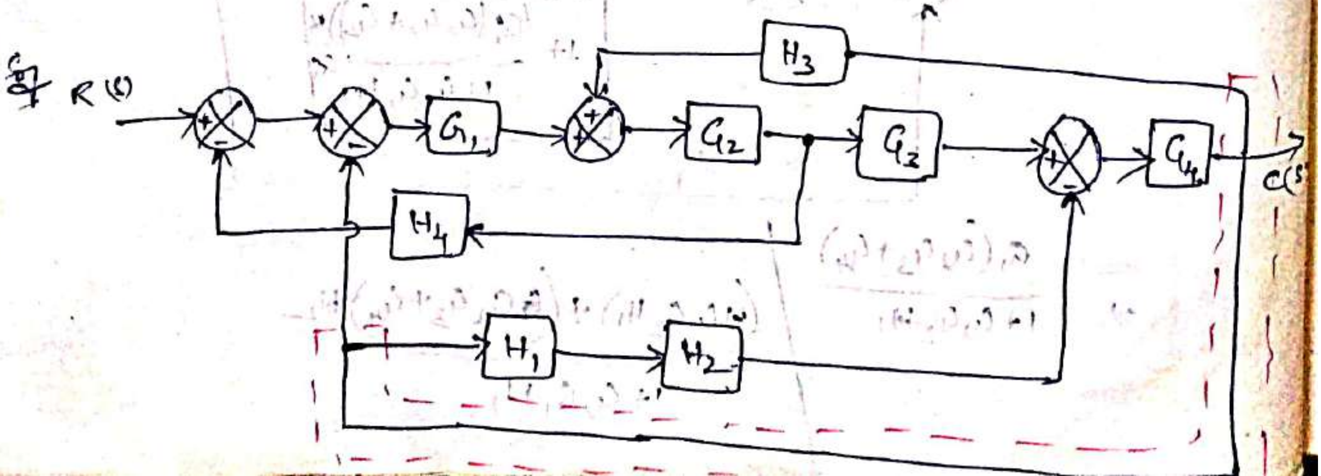


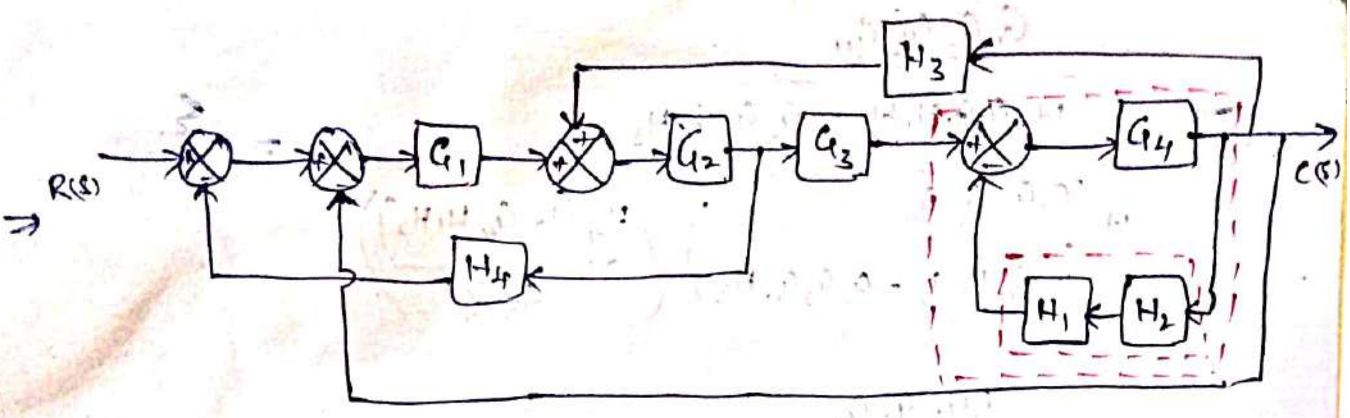
$$\Rightarrow \therefore \frac{C}{R} = \frac{\dfrac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}}{1 + \dfrac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2}}$$

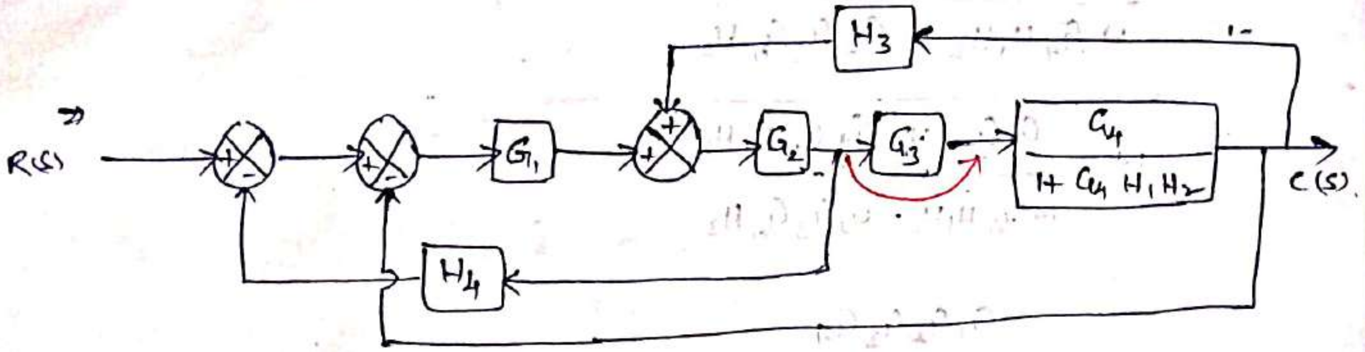
$$\therefore \Rightarrow \boxed{\frac{C}{R} = \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2H_1 + G_2G_3H_2 + G_4H_2 + G_1G_2G_3 + G_1G_4}}$$

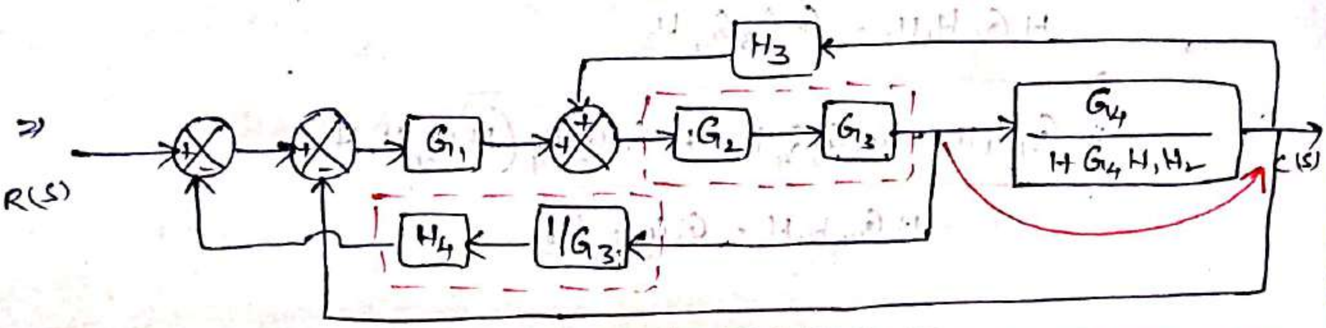
7

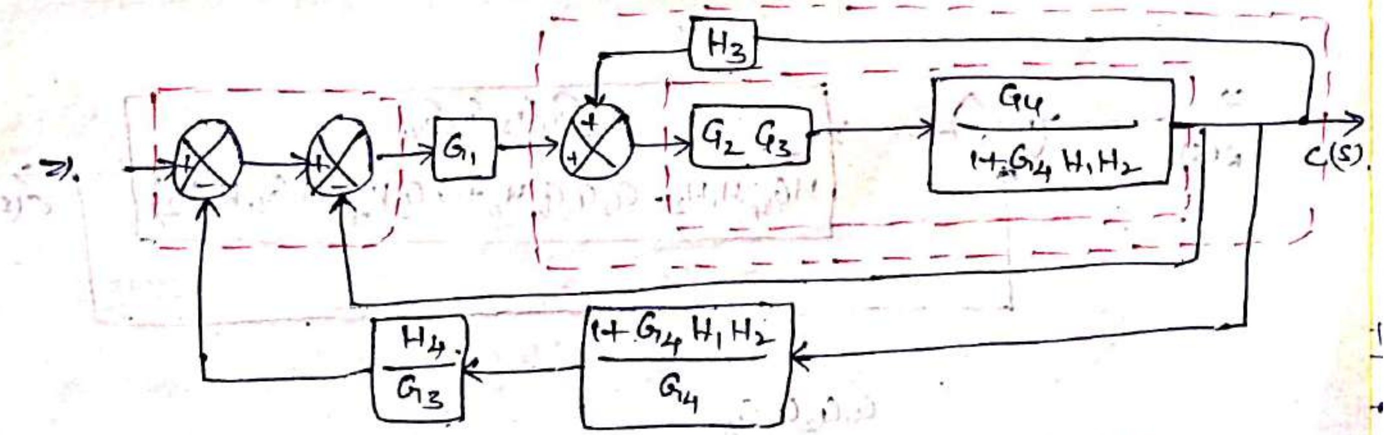


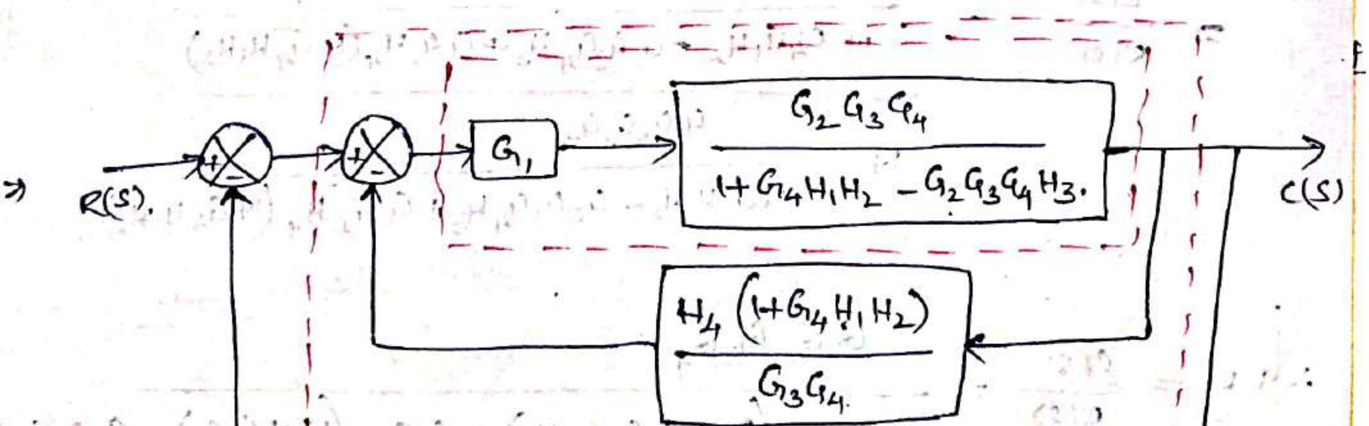



R(s)
G1
G2
G3
G4
H1
H2
H3
H4
C(s)


R(s)
G1
G2
G3
G4 / 1+G4H1H2
H3
H4
C(s)


R(s)
G1
G2
G3
G4 / 1+G4H1H2
H3
H4
1/G3
C(s)


G1
G2 G3
G4 / 1+G4H1H2
H3
H4 / G3
1+G4H1H2 / G4
C(s)

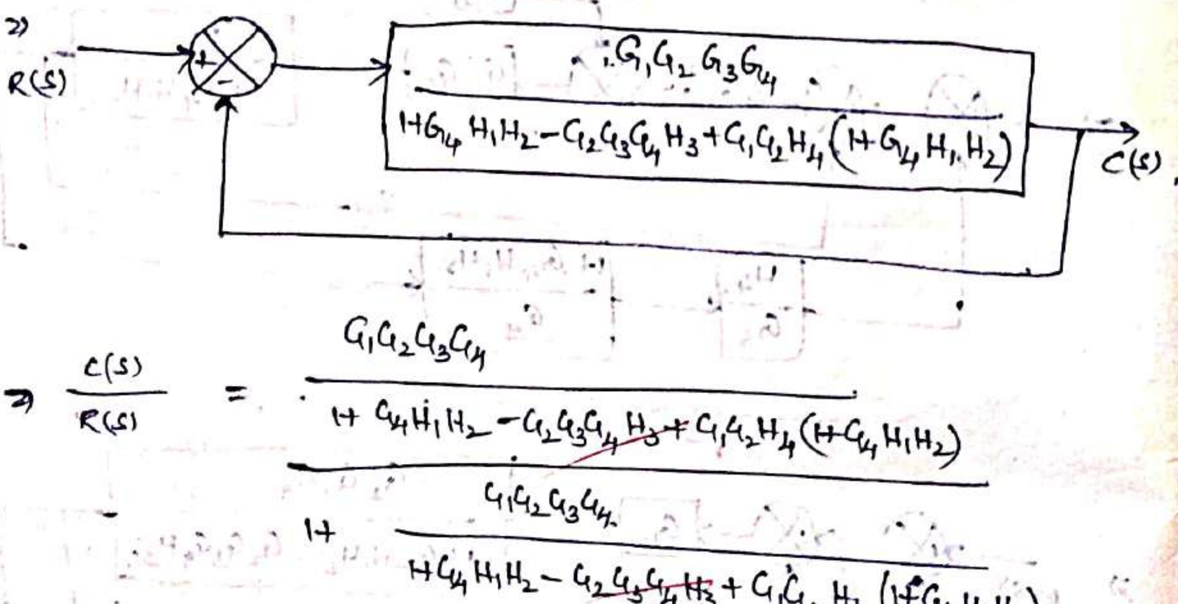

R(s)
G1
G2G3G4 / 1+G4H1H2 − G2G3G4H3
H4(1+G4H1H2) / G3G4
C(s)

$$\Rightarrow \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\left(\dfrac{G_1G_2\cancel{G_3G_4}}{1+G_4H_1H_2-G_2G_3G_4H_3}\right)\left(\dfrac{H_4(1+G_4H_1H_2)}{\cancel{G_3G_4}}\right)} = \frac{C}{R}$$

$$\Rightarrow \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3}}{1+\dfrac{G_1G_2H_4(1+G_4H_1H_2)}{1+G_4H_1H_2-G_2G_3G_4H_3}}$$

$$\Rightarrow \frac{\dfrac{G_1G_2G_3G_4}{\cancel{1+G_4H_1H_2-G_2G_3G_4H_3}}}{\dfrac{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}{\cancel{1+G_4H_1H_2-G_2G_3G_4H_3}}}$$

$\Rightarrow$



$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}{1+\dfrac{G_1G_2G_3G_4}{1+G_4H_1H_2-G_2G_3G_4H_3+G_1G_2H_4(1+G_4H_1H_2)}}$$

$$\therefore \boxed{T.F = \frac{C(S)}{R(S)} = \frac{G_1G_2G_3G_4}{1+H_1H_2(G_4+G_1G_2G_4H_4)+G_1G_2(H_4+G_3G_4)-G_2G_3G_4H_3}}$$

* Signal Flow Graph :-

The signal flow graph is used to Represent the Control System graphically. and it was developed by S.J. MASON.

A signal flow graph is a diagram that represents a set of linear algebraic equations. By taking laplace transform, the time domain differential equations governing a system can be transfered to a set of algebraic equations in s-Domain.

It should be noted that the signal flow graph approach is similar to that of the Block diagram approach. And, this Method (signal flow graph) is simpler than Block diagram approach. Using the Mason Gain Formula the overall gain of the system can be Computed Easily.

* properties of signal flow graph:-

1. The algebraic equations which are used to construct signal flow graph must be in the form of cause and effect Relationship
2. signal flow graph is applicable to linear systems only.
3. A node in the signal flow graph represents the variable (or) signal.
4. A node adds the signals of all incoming branches and transmits the sum to all outgoing branches.
5. A Mixed node which has both incoming and outgoing signals can be treated as output node by adding an outgoing branch of unity transmittance
6. A branch indicates functional dependance of one signal on the other.

7. The signals travel along branches only in the marked direction, and when it travels it gets multiplied by the gain (or) transmittance of the branch.

8. The signal flow graph of system is not unique. By rearranging the system equations different types of signal flow graphs can be drawn for a given system.

* Mason's Gain Formula :-

The Mason's gain formula is used to determine the transfer function of the system from the signal flow graph of the system.

Let, $R(s)$ = Input to system, $C(s)$ = Output of system

$\therefore$ Transfer function $T(s) = \dfrac{C(s)}{R(s)}$.

Mason's Gain formula states the overall gain of system as follows;

Overall gain, $\boxed{T = \dfrac{1}{\Delta} \sum_{K} P_K \Delta_K}$ i.e, $\boxed{T = \dfrac{1}{\Delta} (P_1\Delta_1 + P_2\Delta_2 + \dots)}$

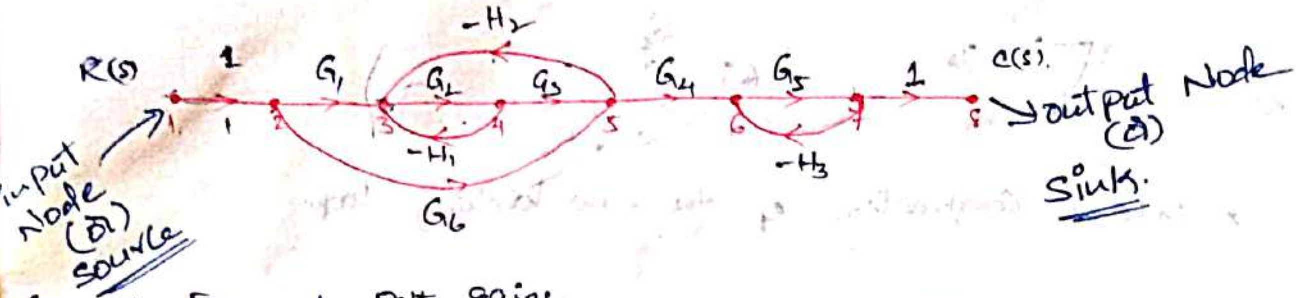
P_K = Forward path gain of K^{th} forward path

Δ = 1 − (Sum of individual loop gains) + (Sum of gain products of all possible Combinations of two non touching loops) − (Sum of gain products of all possible Combinations of 3-non-touching loops) + (. . . . .) . . . .

Δ_K = Δ for the part of the graph which is not touching K^{th} forward path.

Problems:-

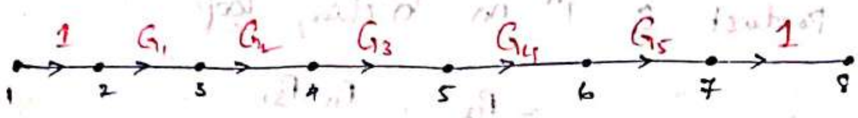
1. Find overall T.F of the System for signal flow graph below.



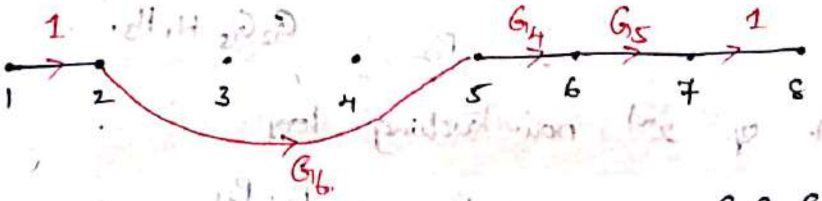
Sol 1. Farward path gains,

There are two farward paths $\therefore K = 2$

let the farward path gains be P_1 and P_2



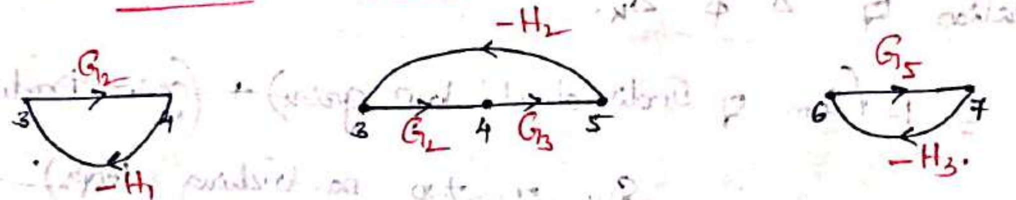
and



Gain for 1st farward path $= P_1 = G_1G_2G_3G_4G_5$.

Gain for 2nd farward path $= G_4G_5G_6 = P_2$.

2. Individual Loop Gains.



I loop-1 I. loop-2 I. loop-3.

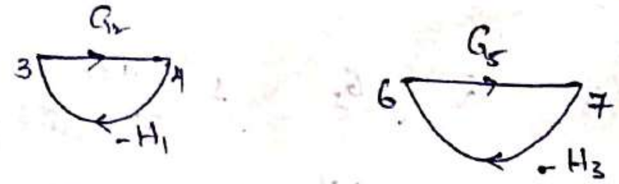
loop gain of Individual loop-1 $= P_{11} = -G_2H_1$

loop gain of Individual loop-2 $= P_{21} = -G_2G_3H_2$

loop gain of Individual loop-3 $= -G_5H_3 = P_{31}$.

3. Gain products of two non-touching loops:-

1. first Combination of two non touching loops.



2. Second Combination of two - non touching loops.

Now, the gain product of 1^{st} non touching loop

$$= P_{12} = P_{11}\, P_{31}$$

$$= (-G_2 H_1)\,(-G_5 H_3)$$

$$\therefore\ P_{12} = G_2 G_5 H_1 H_3.$$

the gain product of 2^{nd} non-touching loop

$$= P_{22} = P_{21}\, P_{31}$$

$$= (-G_2 G_3 H_2)\,(-G_5 H_3)$$

$$\therefore\ P_{22} = G_2 G_3 G_5 H_2 H_3.$$

4. Calculation of Δ & Δ_K.

$\Rightarrow \Delta$ = 1 − (Sum of Individual loop gains) + (gain product of Sum of two non touching loops).

$$\Rightarrow 1 - (P_{11} + P_{21} + P_{31}) + (P_{12} + P_{22})$$

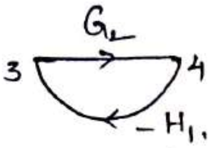
$$= 1 - \left[-G_2 H_1 - G_2 G_3 H_2 + (-G_5 H_3)\right] + (G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3)$$

$$\therefore\ \Delta = 1 + G_2 H_1 + G_2 G_3 H_2 + G_5 H_3 + G_2 G_5 H_1 H_3 + G_2 G_3 G_5 H_2 H_3.$$

$\Delta_1 = 1$, Since there is no part of graph which is non-touching with first forward path.

$\boxed{\text{Always will be } 1}$

Δ_2 = In the 2nd forward path there is one loop which is non-touching to 2nd forward path so, it can be written as



$= 1 - P_{11}$

$= 1 - (-G_2H_1) = 1 + G_2H_1$

[First forward path of there is no non-touching loop. so, $\Delta_1 = 1 - 0 = 1$]

Now,

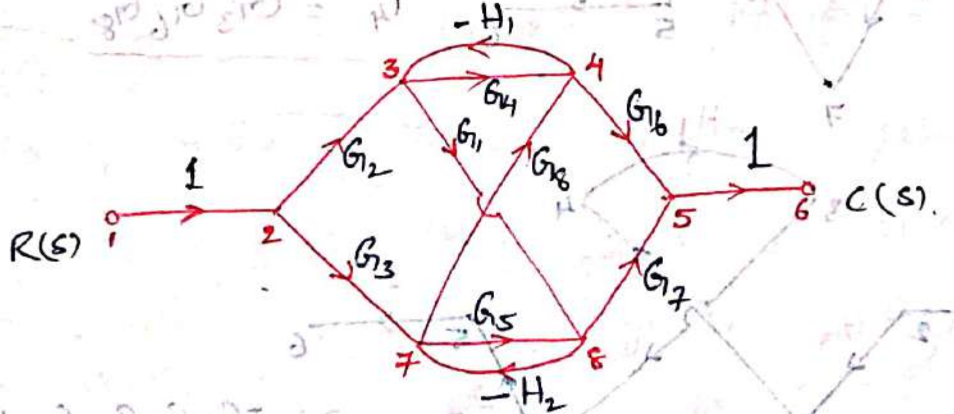
Mason's Gain formula $= \dfrac{1}{\Delta} \sum_K P_K \Delta_K$.

of T.F $= \dfrac{1}{\Delta}\left(P_1\Delta_1 + P_2\Delta_2\right)$

$$\therefore \text{T.F} = \frac{G_1G_2G_3G_4G_5 + G_4G_5G_6\,(1+G_2H_1)}{1 + G_2H_1 + G_3G_2H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}.$$

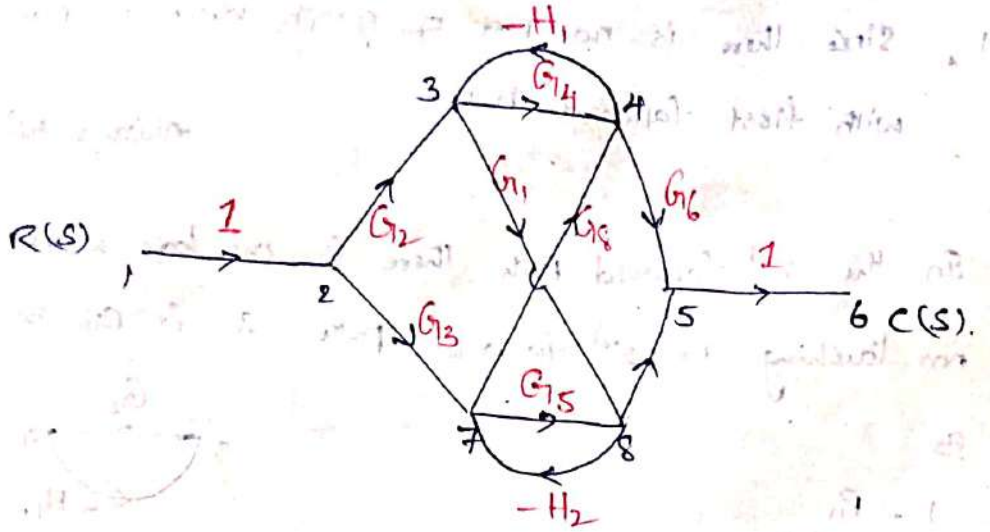
$$\therefore \boxed{\text{T.F} = \frac{G_4G_5\left[G_1G_2G_4 + G_6\left(1+G_2H_1\right)\right]}{1 + G_2H_1 + G_3G_2H_2 + G_5H_3 + G_2G_5H_1H_3 + G_2G_3G_5H_2H_3}}.$$

2.

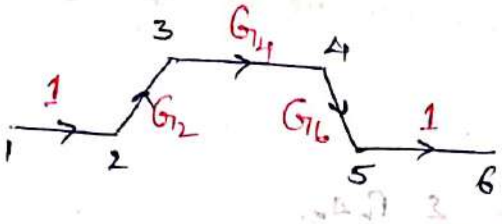


Find the over all gain of Transfer-function for the above Signal flow graph.

Sol.

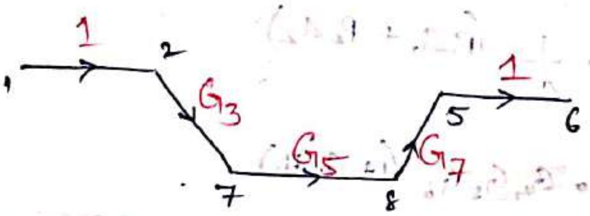


1. The number of forward paths $K = 6$.

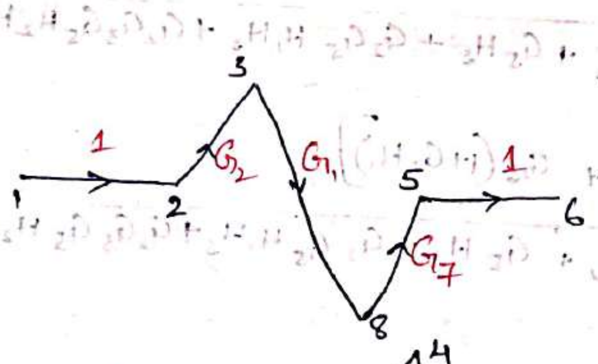


Gain of 1st forward path

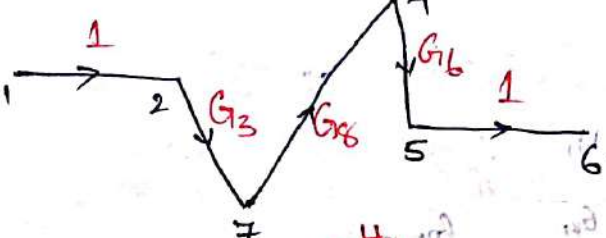
$P_1 = G_2 G_4 G_6$.



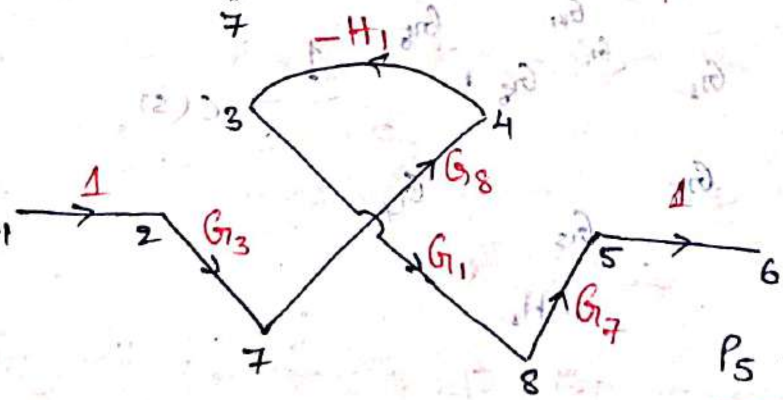
$P_2 = G_3 G_5 G_7$



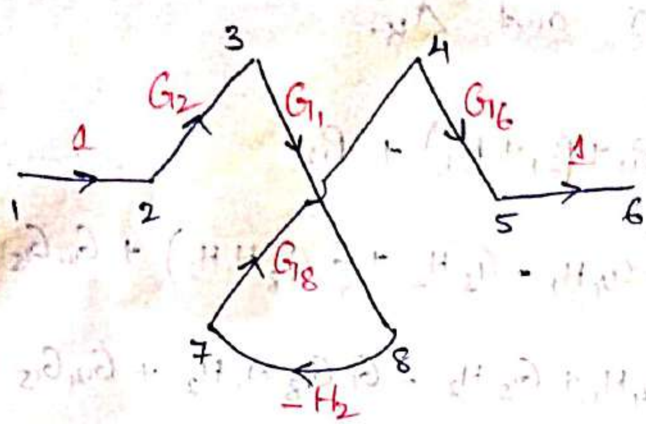
$P_3 = G_1 G_2 G_7$



$P_4 = G_3 G_6 G_8$



$P_5 = -G_1 G_3 G_7 G_8 H_1$.

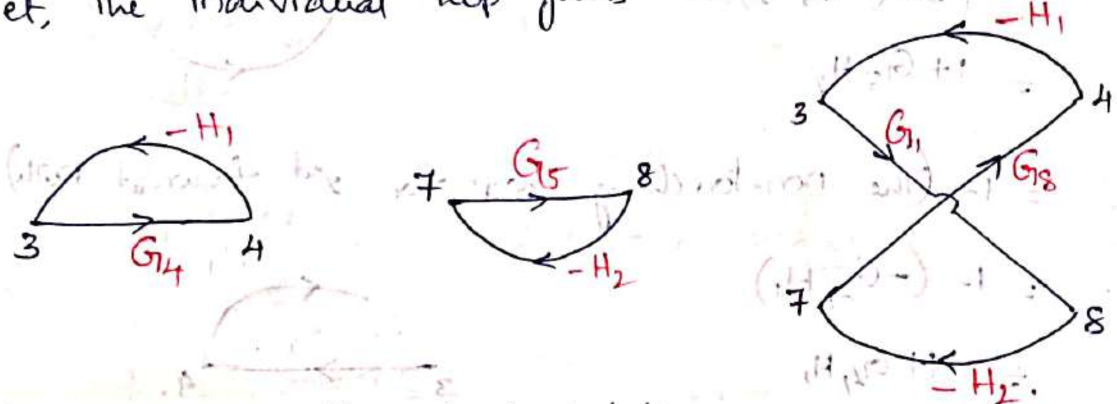


$P_6 = -G_1 G_2 G_{16} G_{18} H_2$.

2. Individual loop Gain.

There are three Individual loops.

let, the individual loop gains be P_{11}, P_{21}, P_{31}.



∴ loop gain of 1st Individual loop

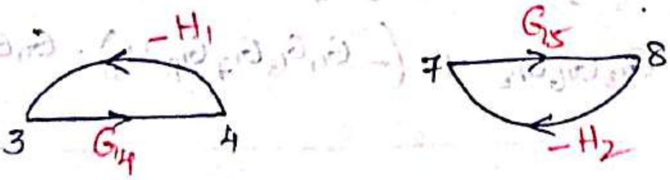
$= P_{11} = -G_{14} H_1$.

loop gain of 2nd Individual loop $= P_{21} = -G_{15} H_2$

loop gain of 3rd Individual loop $= P_{31} = G_1 G_{18} H_1 H_2$.

3: Non-Touching loop :-

There is only one non-touching loop. let the gain product of the non-touching loop be P_{12}.



∴ Gain product of the two non-touching loop be

$$P_{12} = (-G_{14} H_1)(-G_{15} H_2)$$

$$\therefore \; P_{12} = G_{14} G_{15} H_1 H_2$$

4. Calculation of Δ and Δ_K.

$$\Rightarrow \Delta = 1 - (P_{11} + P_{21} + P_{31}) + P_{12}$$

$$= 1 - (-G_4 H_1 - G_5 H_2 + G_1 G_8 H_1 H_2) + G_4 G_5 H_1 H_2$$

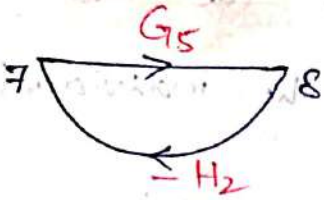
$$\therefore \Delta = 1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2$$

Now,

$\Delta_1 = 1 -$ (The non-touching loop for 1st falward path)

$$= 1 - (-G_5 H_2)$$

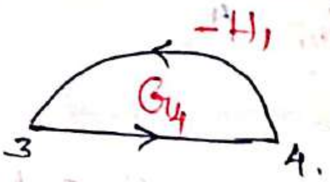
$$= 1 + G_5 H_2$$



$\Delta_2 = 1 -$ (The non-touching loop for 2nd falward path)

$$= 1 - (-G_4 H_1)$$

$$= 1 + G_4 H_1$$



$\Delta_3 = \Delta_4 = \Delta_5 = \Delta_6 = 1.$

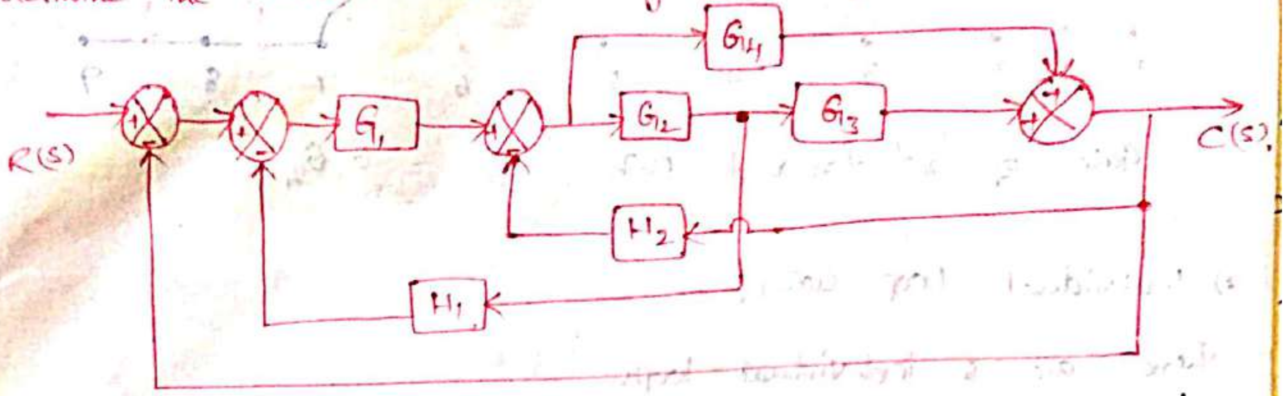
[$\because$ There is no non-touching loop for 3, 4, 5, 6 falward paths, $\therefore$ taken as 1]

5. Transfer function using Mason's gain formula,

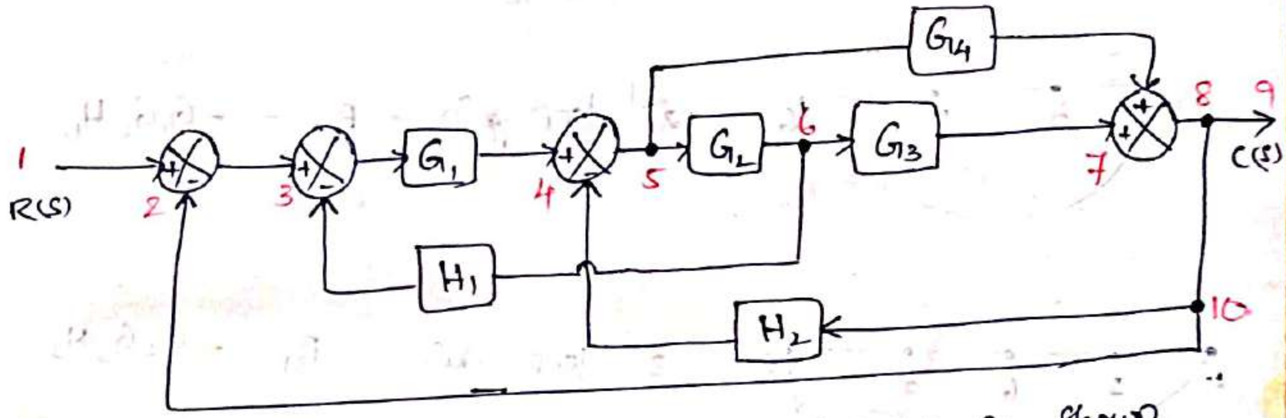
$$T = \frac{1}{\Delta}\left[\sum_K P_K \Delta_K\right] = \frac{1}{\Delta}\left[P_1\Delta_1 + P_2\Delta_2 + P_3\Delta_3 + P_4\Delta_4 + P_5\Delta_5 + P_6\Delta_6\right]$$

$$\Rightarrow T = \frac{G_2 G_4 G_6 (1 + G_5 H_2) + G_3 G_5 G_7 (1 + G_4 H_1) + G_1 G_2 G_7 + G_3 G_6 G_8 + (-G_1 G_3 G_7 G_8 H_1) - G_1 G_2 G_6 G_8 H_2}{1 + G_4 H_1 + G_5 H_2 - G_1 G_8 H_1 H_2 + G_4 G_5 H_1 H_2}$$

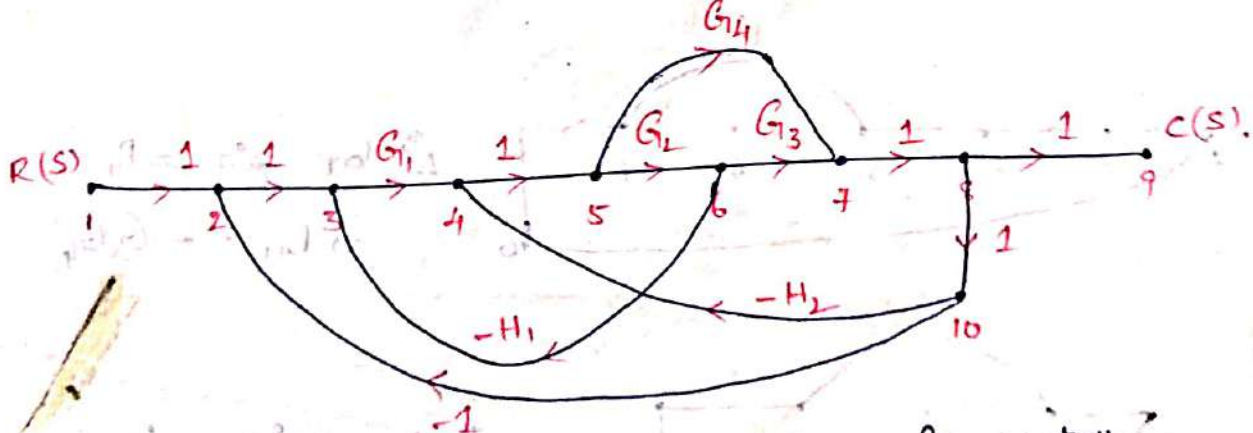
3- Convert the block diagram to signal flow graph and to determine the transfer function using Mason's gain formula.



Sol. Firstly, the Nodes are assigned at every branch node and at the summing points of block diagram as shown below



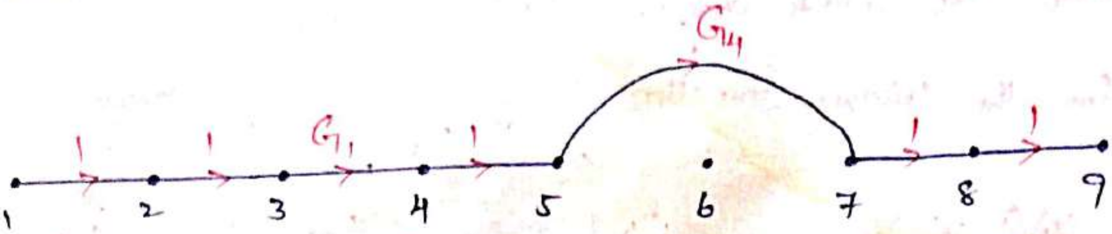
The signal flow graph for above block diagram as shown.



1) From the signal flow graph, there are two forward paths

$\therefore K = 2$

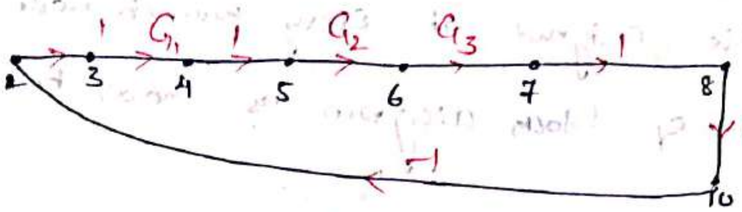
$\therefore$ Gain of 1st forward path $= P_1 = G_1 G_2 G_3$.



Gain of 2^{nd} forward path $= P_2 = G_1 G_4$.

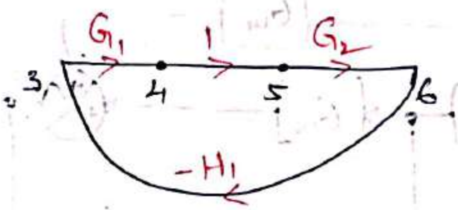
2) Individual Loop Gain :-

There are 5 individual loops.

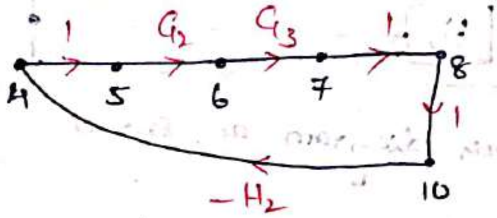


1^{st} loop gain =

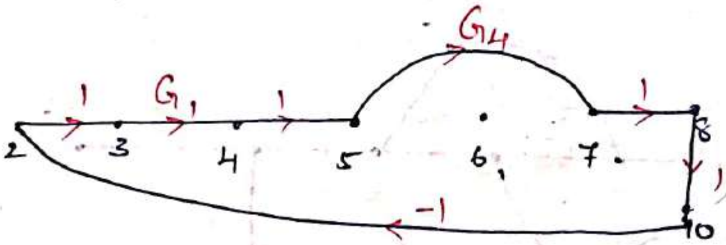
$P_{11} = -G_1 G_2 G_3$.



2^{nd} loop gain $= P_{21} = -G_1 G_2 H_1$.

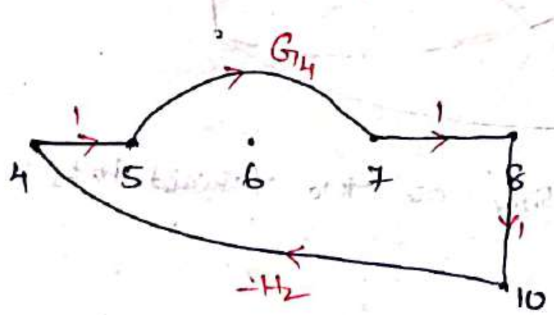


3^{rd} loop gain $= P_{31} = -G_2 G_3 H_2$



4^{th} loop gain $= P_4$

$\Rightarrow P_{41} = -G_1 G_4$



5^{th} loop gain $= P_5$

$\Rightarrow P_{51} = -G_4 H_2$

3. Non-Touching loops :-

There is no possibility of two non-touching loops from the signal flow graph.

4. Calculation of Δ and Δ_K:-

$$\Delta = 1 - (P_{11} + P_{21} + P_{31} + P_{41} + P_{51})$$

$$\Delta = 1 - (-G_2G_1G_3 - G_1G_2H_1 - G_2G_3H_2 - G_1G_4 - G_4H_2)$$

$$\therefore \Delta = 1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2.$$

Now, there is no non-touching loops for the forward paths 1 and 2 $\therefore$ $\Delta_1 = \Delta_2 = 1.$

5. Transfer function using Mason's gain formula :

$$T = \frac{1}{\Delta} \sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

$$= \frac{G_1G_2G_3 + G_1G_4}{1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2}$$

$$\therefore \boxed{T = \frac{G_1(G_2G_3 + G_4)}{1 + G_1G_2G_3 + G_1G_2H_1 + G_2G_3H_2 + G_1G_4 + G_4H_2}.}$$

4. Construct signal flow graph for set of Equations and find overall gain using Mason's gain formula.

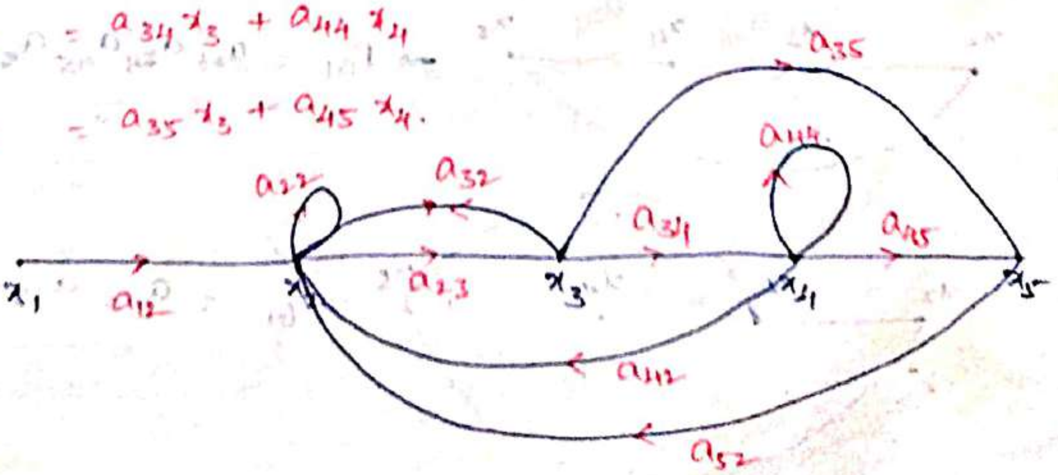
$$x_2 = a_{12}x_1 + a_{22}x_2 + a_{32}x_3 + a_{42}x_4 + a_{52}x_5$$

$$x_3 = a_{23}x_2$$

$$x_4 = a_{34}x_3 + a_{44}x_4$$

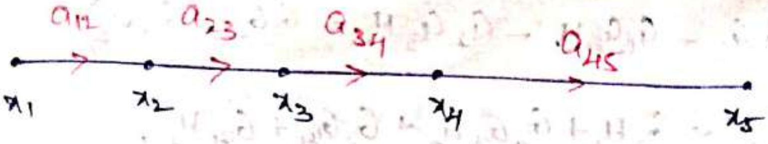
$$x_5 = a_{35}x_3 + a_{45}x_4.$$

Sol:

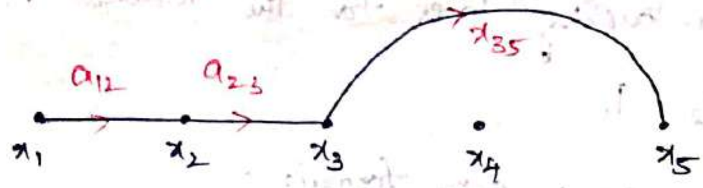


1. Farward paths.

There are two farward paths $\Rightarrow K = 2$



$\Rightarrow P_1 = a_{12}a_{23}a_{34}a_{45}$



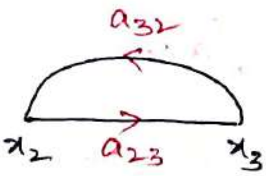
$\therefore$ The loop gain for 2^{nd} farward Path $\Rightarrow P_2 = a_{12}a_{23}a_{35}$.

2. Individual Loops:-

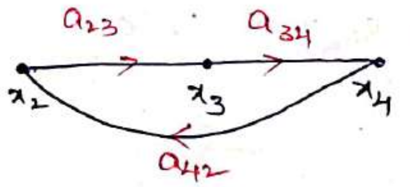
There are totally 6 individual loops, They are.

$\Rightarrow P_{11} = a_{22}$

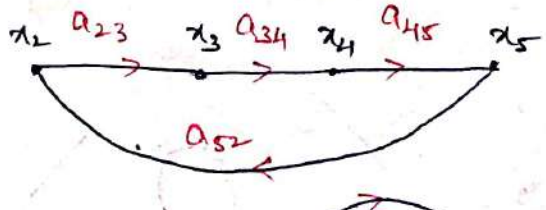
$\Rightarrow P_{21} = a_{44}$



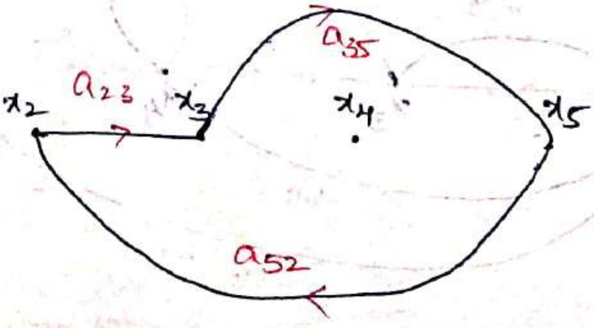
$\Rightarrow P_{31} = a_{23}a_{32}$



$\Rightarrow P_{41} = a_{23}a_{34}a_{42}$



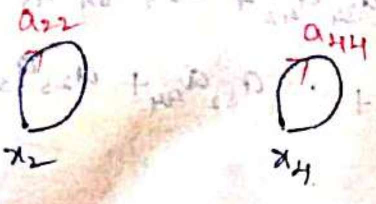
$\Rightarrow P_{51} = a_{23}a_{34}a_{45}a_{52}$



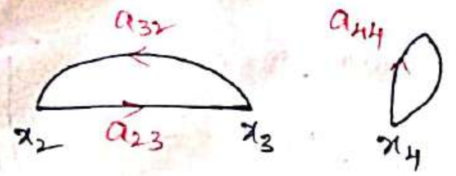
$\Rightarrow P_{61} = a_{23}a_{35}a_{52}$

3. Non-touching loops:

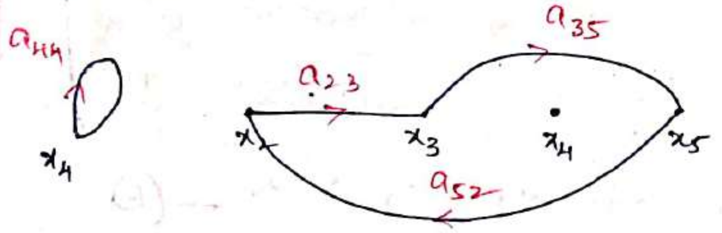
There are 3 non-touching loop pairs exists. They are,



$\Rightarrow P_{12} = a_{22}a_{44}$ [$\because$ gain of two non-touching loops.]



$\Rightarrow P_{22} = a_{23}a_{32}a_{44}$.



$\Rightarrow P_{32} = a_{23}a_{35}a_{52}a_{44}$.

4. Calculation of Δ and Δ_K :

$\Delta = 1 -$ (Sum of Individual loops) + (Sum of gain product of two non touching loops)

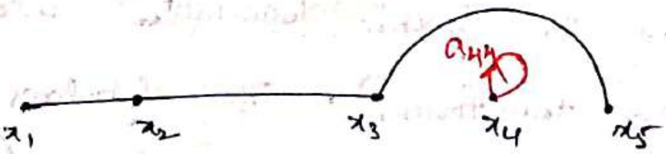
$$= 1 - \left(a_{22}+a_{44}+a_{23}a_{32}+a_{23}a_{34}a_{42}+a_{23}a_{34}a_{52}a_{45}+ a_{23}a_{35}a_{52}\right) + \left(a_{22}a_{44}+a_{23}a_{32}a_{44}+a_{23}a_{35}a_{52}\right)$$

$\therefore \Delta_1 = 1 -$ (non-touching loops in forward path-1)

$= 1-0 = 1.$

$\Delta_2 = 1 -$ (non-touching loops in forward path-2)

$= 1 - a_{44}$.



5. Transfer function using Mason's gain formula:

$$T = \frac{1}{\Delta}\sum_K P_K \Delta_K$$

$$= \frac{1}{\Delta}\left(P_1\Delta_1 + P_2\Delta_2\right).$$

$$\Rightarrow T = \frac{a_{12}a_{23}a_{34}a_{45} + a_{12}a_{23}a_{35}(1-a_{44})}{1-(a_{22}+a_{44}+a_{23}a_{32}+a_{23}a_{34}a_{42}+a_{23}a_{34}a_{45}a_{52}+a_{23}a_{35}a_{52})+(a_{22}a_{44}+a_{23}a_{32}a_{44}+a_{23}a_{35}a_{52})}$$

* Comparison between A.C and D.C. Servomotors.
* Comparison between Armature Controlled and Field Controlled D.C. Servomotors.
* Comparison between Block Diagram & signal flow graph.

(A)

* Comparison between open loop and closed loop Systems — (B)

Sol. (A) → U.A. Bakshi and V.U. Bakshi (Control Systems)

(B) → A. Nagoor Kani. (Control Systems).

* Synchros :-

Synchros are widely used in control systems as detectors and encoders because of their high Reliability. Synchro is basically an electromagnetic ~~device~~ transducer, which converts the angular position of the shaft into Electric signal.

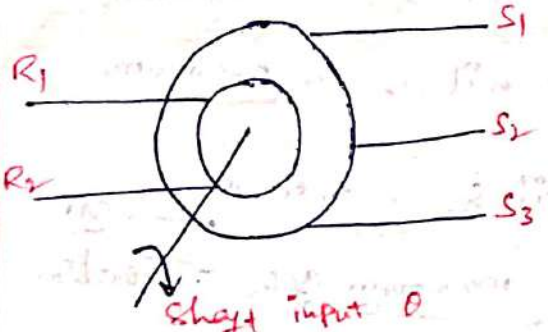
A Synchro system is formed by interconnection of devices called the Synchro transmitter and Synchro Reciever (Synchro Control transformer). The Synchros is basically works on the principle of Rotating transformer (Induction motor). The Synchro pair (transmitter & Reciever) measures and compares two angular displacements. They can be used in following two ways.

1. To Control the angular position of load from a remote place/ long distance.

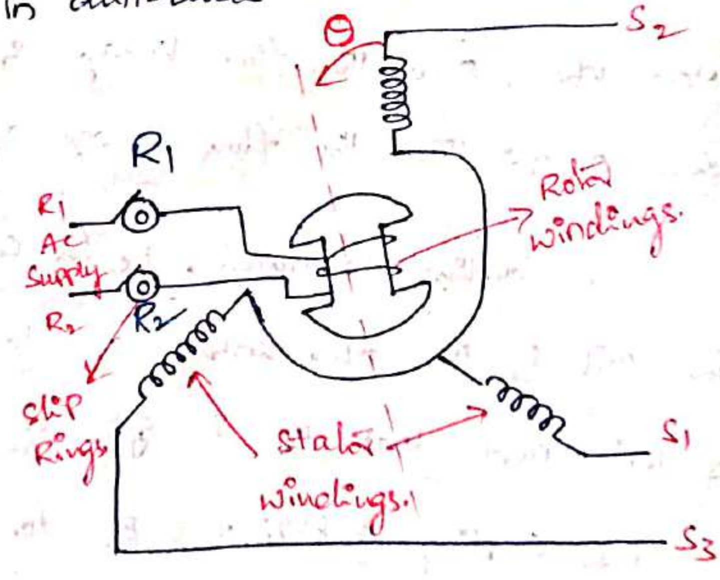
2. For automatic correction of changes due to disturbance in the angular position of the load.

Synchro Transmitter :-

The Constructional features, electrical circuit and a schematic symbol of synchro transmitter as shown below. The two major parts of synchro transmitter are stator and Rotor. The stator is identical to the stator of three phase alternator. It is made up of laminated steel and slotted on the inner periphery to achieve the balanced 3ϕ winding. The stator wdg is of concentric coil type with the axis of 3 coils 120° apart. The stator winding is star connected. Assummed rotor rotates in anti-clock wise direction.



Schematic Symbol



Electric circuit

The Rotor is of Dumb bell shape in construction with a single winding. A 1ϕ AC excitation voltage is applied to the Rotor through the slip Rings

[once. Refer A. Nagoor Kani text book page - 172 you can understand in clear]

→ The ac voltage applied to rotor = $e_r = E_r \sin\omega t$ —(1)

⇒ Induced Emf in stator coil = $K_t K_c E_r \sin\omega t$ —(2)

where, e_r = Voltage (ac) to rotor

E_r = Maximum value of rotor Excitation voltage

ω = Angular frequency

K_t = Turns ratio of stator & Rotor windings.

K_c = Coupling Coefficient

θ = Angular displacement of rotor w.r.t. Reference

Here, the S_2 is taken as Reference and 'θ' is the angular displacement of rotor w.r.t. Reference, Hence, the Coupling coefficient is expressed as,

$K_c = K_1 \cos\theta$ —(3) ⇒ Coupling Coefficient is defined, which is the flux linkage of coil is function of $\cos\theta$.

Now, when $\theta = 0°$ the flux linkage of coil S_2 is maximum.

when $\theta = 90°$ then the flux linkage of coil S_2 is minimum.

∴ Coupling coefficient, K_c for coil-S_2 = $K_1 \cos\theta$ —(4)

for, coil S_3 then flux linkage will be maximum after a rotation of 120° and that of S_1 after a rotation of 240° in anti-clock wise direction

i.e, Coupling coefficient, K_c for coil-S_3 = $K_1 \cos(\theta - 120°)$ —(5)

Coupling coefficient, K_c for coil S_1 = $K_1 \cos(\theta - 240°)$ —(6)

Hence, the emf's of stator coils w.r.t. neutral

$$\left.\begin{aligned} e_{S_2} &= K_t K_1 \cos\theta\, E_r \sin\omega t = K E_r \cos\theta \sin\omega t \\ e_{S_3} &= K_t K_1 \cos(\theta-120°) E_r \sin\omega t = K E_r \cos(\theta - 120°) \sin\omega t \\ e_{S_1} &= K_t K_1 \cos(\theta - 240) E_r \sin\omega t = K E_r \cos(\theta - 240°) \sin\omega t \end{aligned}\right\} \text{—(7)}$$

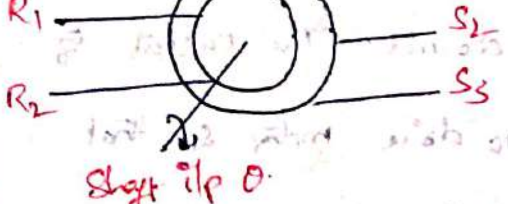
Now, the coil-to-coil emf can be expressed by KVL from the figure below.

$$\left.\begin{aligned} e_{s_1}e_{s_2} &= e_{s_1}-e_{s_2} = \sqrt{3}\,KE_r \sin(\theta+240^\circ)\sin\omega t \\ e_{s_2}e_{s_3} &= e_{s_2}-e_{s_3} = \sqrt{3}\,KE_r \sin(\theta+120^\circ)\sin\omega t \\ e_{s_3}e_{s_1} &= e_{s_3}-e_{s_1} = \sqrt{3}\,KE_r \sin\theta \sin\omega t \end{aligned}\right\} \quad (8)$$

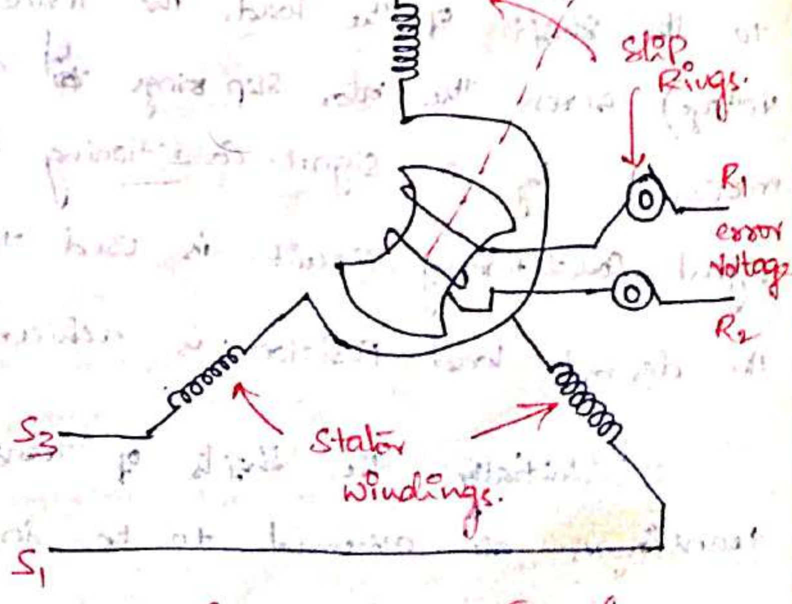
It is observed that when $\theta = 0^\circ$ then the $e_{s_3}e_{s_1} = 0$, and this position is called (or) defined as "Electrical zero of the transmitter."

Synchro Reciever :-

The constructional features of synchro reciever is similar to that of synchro transmitter, except the shape of rotor. The rotor of control transformer is made cylindrical so that the air gap is practically uniform. This feature of control transformer minimizes the changes in the rotor impedance with the rotation of the shaft. The constructional features, electrical circuit and a schematic symbol of control transformer are shown below.



Schematic Symbol

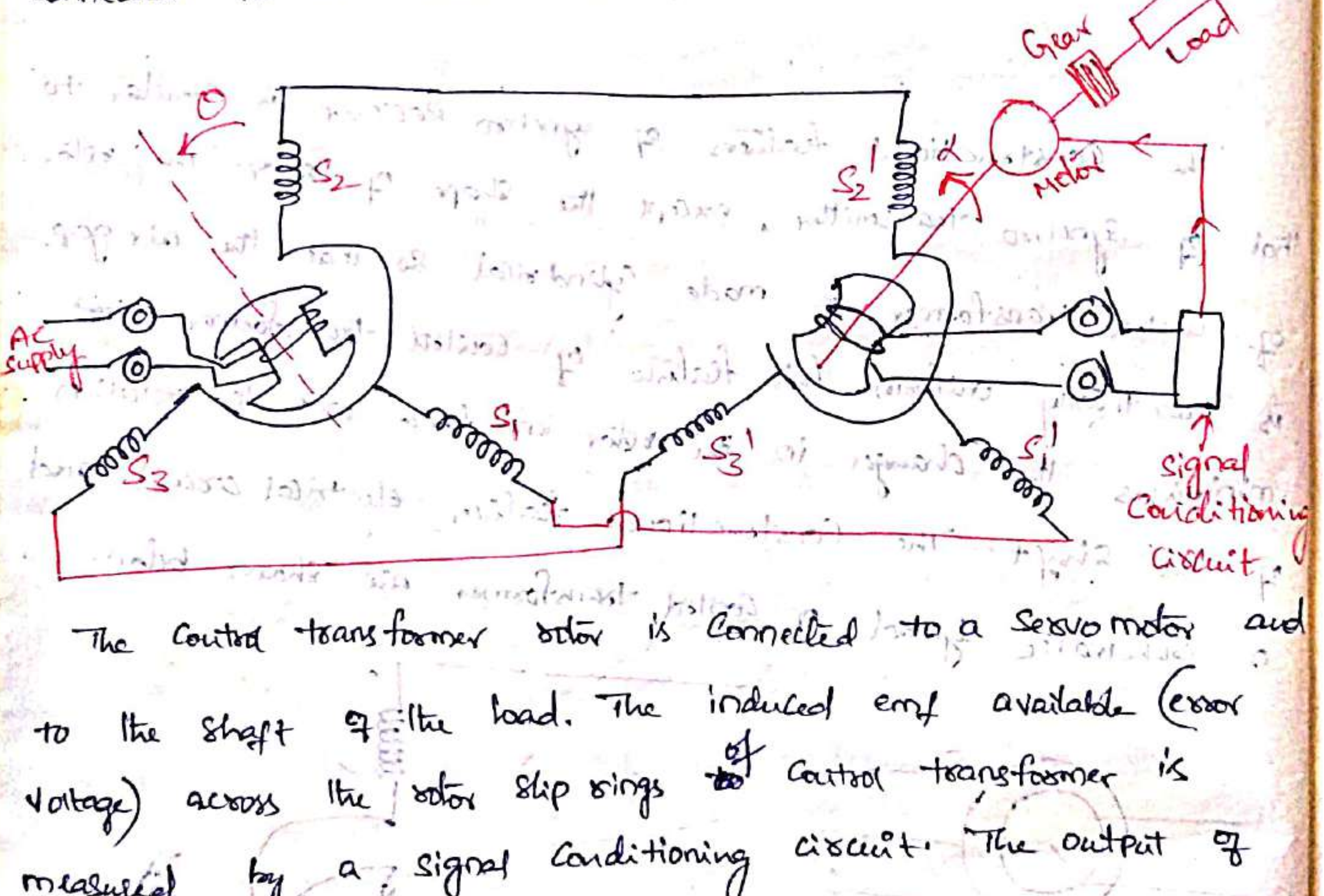


Synchro Control Transformer

The generated emf of the transmitter is applied as input to the stator coils of control transformer. The Rotor shaft is connected to the load whose position has to be maintained at the desired value.

* Synchro As Error Detector :-

The synchro error detector is formed by interconnection of a synchro transmitter and synchro control transformer. In this connection the stator leads of transmitter is directly connected to the stator leads of the control Transformer.



The control transformer rotor is connected to a servo motor and to the shaft of the load. The induced emf available (error voltage) across the rotor slip rings of control transformer is measured by a signal conditioning circuit. The output of signal conditioning circuit is used to drive motor so that the desired load position is achieved.

Initially, the shafts of transmitter and control transformer are assumed to be in aligned position. In this position the transmitter rotor will be in electrical zero position and the control transformer rotor will be in null position. And here the rotors are aligned in 90° to each other.

When the transmitter is excited, the stator flux is set up and emf's are induced in stator coils. The currents in the stator coils set up flux in control transformer.

Let the rotor of transmitter rotate through an angle θ from its electrical zero position. Now the rotor of control transformer will rotate in same direction through an angle α. The net angular separation of the two rotors is equal to $(90-\theta+\alpha)$, and the voltage induced in control transformer rotor is proportional to cosine of this angle

∴ Voltage across slip rings of C.T = e_m

$$\Rightarrow e_m = K \cdot E_r \cos(90-\theta+\alpha) \sin\omega t$$

$$= K \cdot E_r \cos(90-(\theta-\alpha)) \sin\omega t$$

$$\therefore e_m = K \cdot E_r \cdot \sin(\theta-\alpha) \sin\omega t$$

where, K is proportionality constant.

let, $\theta-\alpha = \phi(t)$

for small values, $\phi(t) = \sin(\theta-\alpha) = \sin\phi(t) = \phi(t)$.

∴ Modulated error voltage $e_m = K \cdot E_r \sin\phi(t) \sin\omega t$.

Now, the error voltage again demodulates by the signal conditioning circuit to drive the motor is given by,

demodulated error voltage, $e = K_s \cdot \phi(t)$.

K_s = Sensitivity of synchro error detector in volts/deg.

⇒ On taking laplace transform,

$$E(s) = K_s \phi(s)$$

$$\Rightarrow \boxed{\frac{E(s)}{\phi(s)} = K_s}$$ which is the T.F of Synchro Error Detector.

* Transfer function of Synchro transmitter - Reciever.

let, θ = angle of rotation of transmitter from electrical zero position.

α = angle of rotation of Reciever from electrical Null position.

Then, the torque Produced by the Reciever is given by,

$$T_r(t) = K[\theta(t) - \alpha(t)] \quad \text{--- (1)}$$

applying laplace transform,

$$T_r(s) = K[\theta(s) - \alpha(s)] \quad \text{--- (2)}$$

K = Sensitivity of Error Detector

let $\delta(s) = [\theta(s) - \alpha(s)]$

$$\therefore T_r(s) = K \cdot [\delta(s)] \quad \text{--- (3)}$$

If J Represents movement of inertia, B is frictional Coefficient of Synchro Control transformer, then Torque developed is given by,

$$T_r(t) = J\frac{d^2\alpha(t)}{dt^2} + B\frac{d\alpha(t)}{dt} \quad \text{--- (4)}$$

$$\Rightarrow T_r(s) = Js^2\alpha(s) + Bs\alpha(s) \quad \text{--- (5)}$$

At Equilibrium (2) is equal to (5)

$$\Rightarrow K[\theta(s) - \alpha(s)] = Js^2\alpha(s) + Bs\alpha(s)$$

$$\Rightarrow K \cdot \theta(s) = Js^2\alpha(s) + Bs\alpha(s) + K\cdot\alpha(s).$$

$$= \alpha(s)[Js^2 + Bs + K].$$

∴ The transfer function $= \dfrac{\alpha(S)}{\theta(S)}$.

$$\Rightarrow \boxed{\frac{\alpha(S)}{\theta(S)} = \frac{K}{JS^2 + BS + K}}$$

The above equation is called as the Transfer function of Synchro transmitter and Reciever.

$$e_{s_1 s_2} = e_{s_1} - e_{s_2} = KE_r \sin\omega t \left[\cos(240+\theta) - \cos\theta\right]$$

$$= KE_r \sin\omega t \left[\cos\theta \cos 240^\circ + \sin\theta \sin 240^\circ - \cos\theta\right]$$

$$= KE_r \sin\omega t \left[\cos\theta\left(\frac{-1}{2}\right) + \sin\theta\left(\frac{-\sqrt{3}}{2}\right) - \cos\theta\right]$$

$$= KE_r \sin\omega t \left(\cos\theta\left(\frac{-3}{2}\right) + \sin\theta\left(\frac{-\sqrt{3}}{2}\right)\right)$$

$$= \sqrt{3} KE_r \sin\omega t \left[\cos\theta\left(\frac{-\sqrt{3}}{2}\right) + \sin\theta\left(\frac{-1}{2}\right)\right]$$

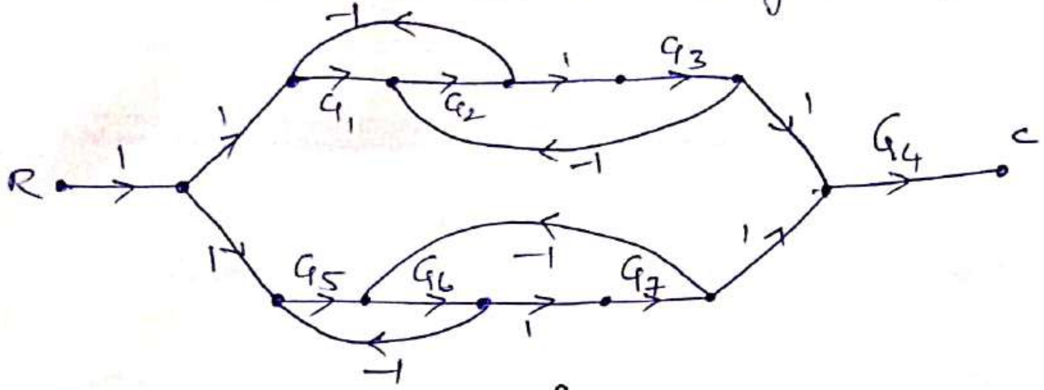
$$= \sqrt{3} KE_r \sin\omega t \left(\cos\theta \sin 240^\circ + \sin\theta \cos 240^\circ\right)$$

$$= \sqrt{3} KE_r \sin\omega t \left[\sin(\theta + 240^\circ)\right]$$

Similarly, $e_{s_2 s_3} = \sqrt{3} KE_r \sin\omega t \sin(\theta + 120^\circ)$

$e_{s_3 s_1} = \sqrt{3} KE_r \sin\omega t \sin\theta$

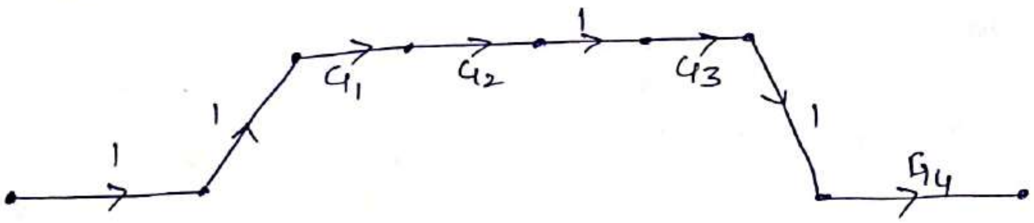
1. Find the Transfer function for the given signal flow graph.



Sol. Step-1: forward path gains.

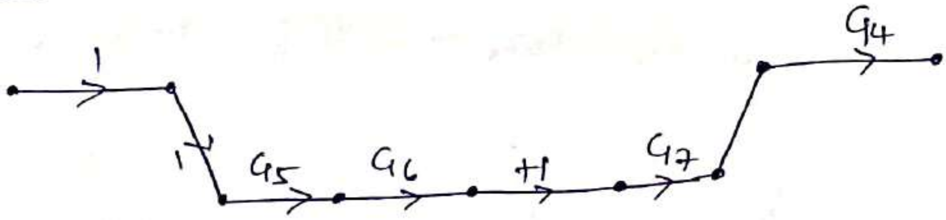
There are two forward paths ∴ $K = 2$

let the forward path gains be P_1 and P_2



$P_1 = G_1G_2G_3G_4$

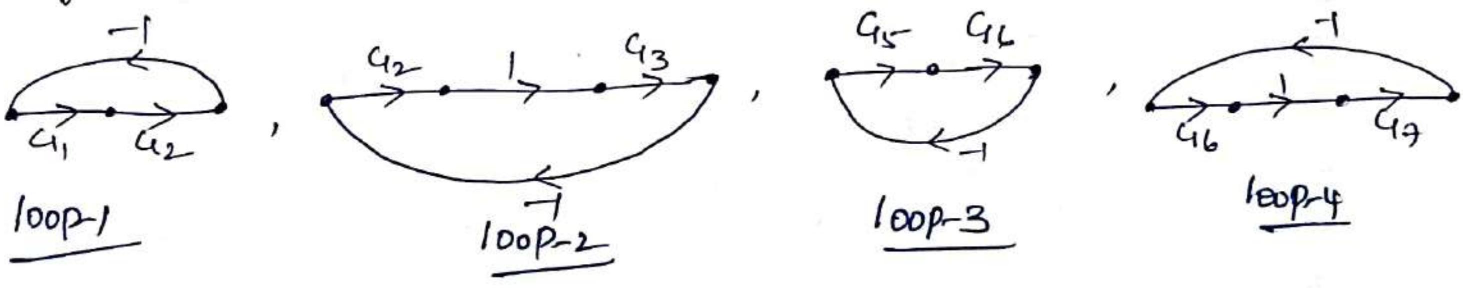
second forward path:



$P_2 = G_4G_5G_6G_7$

Step-2: To find Individual loop gains:

Totally 4 individual loops are there. They are shown below,



gain of individual loop-1 $\Rightarrow P_{11} = -G_1G_2$

" " loop-2 $\Rightarrow P_{21} = -G_2G_3$

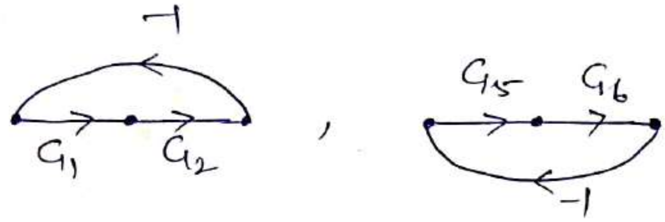
" " loop-3 $\Rightarrow P_{31} = -G_5G_6$

" " loop-4 $\Rightarrow P_{41} = -G_6G_7$

Step-3 :- Finding gain of two Non-touching loops. ②

There are 4 Combinations of two-Non-touching loops are there, They are,

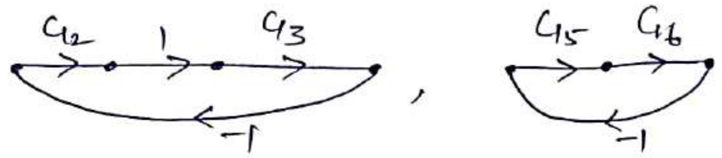
1.



$\Rightarrow P_{12} = (P_{11} \times P_{31})$

gain product of 1st two - Non-touching loops $\Rightarrow P_{12} = G_1G_2G_5G_6$

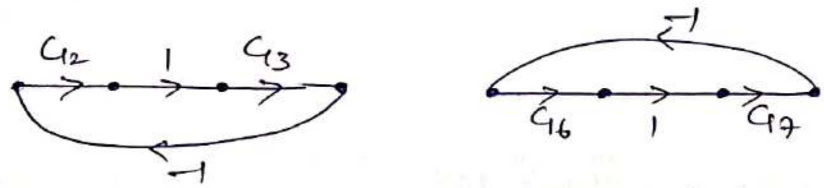
2.



$\Rightarrow P_{22} = P_{21} \times P_{31}$

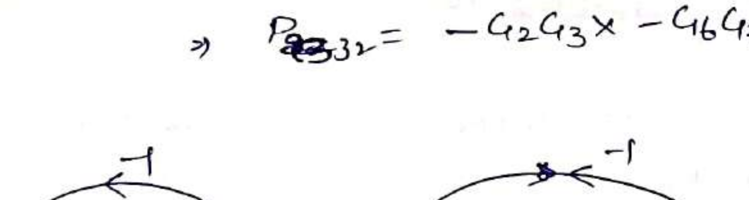
$\Rightarrow P_{22} = G_2G_3G_5G_6.$

3.



$\Rightarrow P_{32} = -G_2G_3 \times -G_6G_7 = G_2G_3G_6G_7$

4.



$P_{42} = -G_1G_2 \times -G_6G_7$

$\therefore P_{42} = G_1G_2G_6G_7$

Step-4 :- Calculation of Δ and Δ_K.

$\Delta = 1 - (\text{Sum of Individual loop gains}) + (\text{gain product of two Non touching loops})$

$- (\text{3 Non-touching loops}) + \cdots$

$= 1 - (P_{11} + P_{21} + P_{31} + P_{41}) + (P_{12} + P_{22} + P_{32} + P_{42})$

$\therefore \Delta = 1 + (G_1G_2 + G_2G_3 + G_5G_6 + G_6G_7) + (G_1G_2G_5G_6 + G_2G_3G_5G_6 + G_2G_3G_6G_7 + G_1G_2G_6G_7)$

Δ_1 = In first forward path there are 2 two loops which are not touching to first forward path, So, can be written as,

$$= 1 - (-G_5G_6 - G_6G_7)$$

$$= (1 + G_5G_6 + G_6G_7)$$

Similarly, $\Delta_2 = 1 - (-G_1G_2 - G_2G_3)$

$$= (1 + G_1G_2 + G_2G_3).$$

Now, using Mason's gain formula,

the T.F $= \dfrac{1}{\Delta} \sum_{K=1}^{2} P_K \Delta_K$

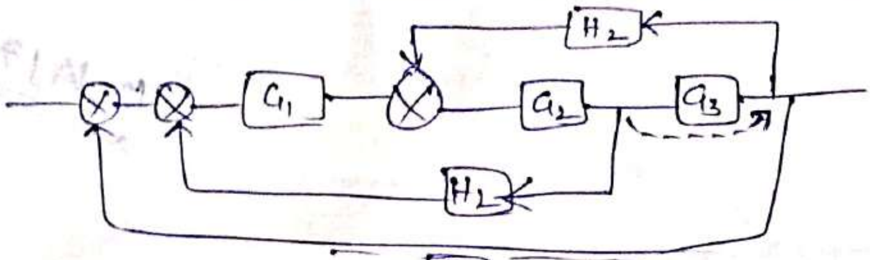
$$= \frac{1}{\Delta}(P_1\Delta_1 + P_2\Delta_2)$$

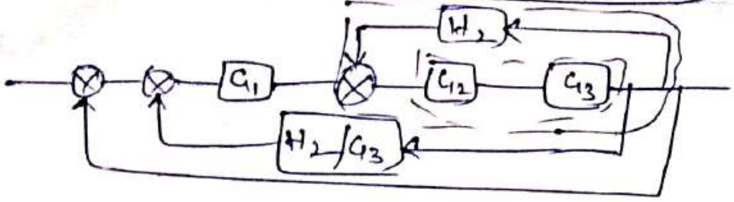
$$= \frac{\left[(G_1G_2G_3G_4)(1 + G_5G_6 + G_6G_7)\right] + \left[(G_4G_5G_6G_7)(1 + G_1G_2 + G_2G_3)\right]}{\Delta}$$

∴ Overall Transfer function will be,

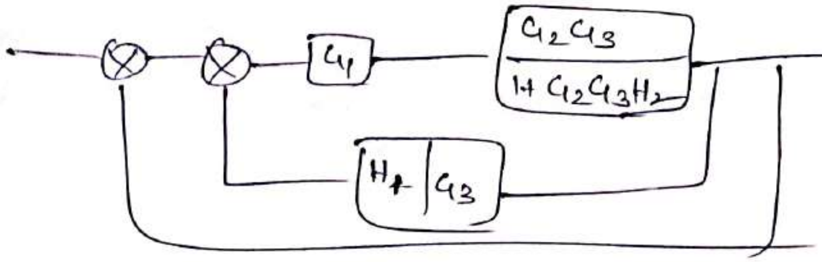
$$T.F = \frac{\left[(G_1G_2G_3G_4)(1 + G_5G_6 + G_6G_7)\right] + \left[(G_4G_5G_6G_7)(1 + G_1G_2 + G_2G_3)\right]}{1 + (G_1G_2 + G_2G_3 + \cancel{G_3G_4} + G_6G_5 + G_6G_7) + \left[(G_1G_2G_5G_6) + (G_2G_3G_5G_6) + (G_2G_3G_6G_7) + (G_1G_2G_6G_7)\right]}.$$

(4)



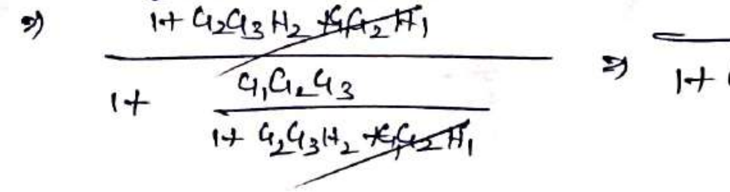


$$\Rightarrow \frac{G_2G_3}{1+G_2G_3H_2}$$

$\Rightarrow$ 

$$\Rightarrow \frac{\frac{G_1G_2G_3}{1+G_2G_3H_2}}{1+\frac{G_1G_2\cancel{G_3}}{1+G_2G_3H_2}\times\frac{H_1}{\cancel{G_3}}} \Rightarrow \frac{\frac{\cancel{G_1G_2G_3}}{1+\cancel{G_2G_3H_2}}}{\frac{(1+G_2G_3H_2)+G_2G_3\times(H_1/G_3)}{1+G_2G_3H_2}} \Rightarrow \frac{\frac{G_1G_2G_3}{\cancel{1+G_2G_3H_2}}}{\frac{1+G_2G_3H_2+G_1G_2H_1}{\cancel{1+G_2G_3H_2}}}$$

$$\Rightarrow \frac{G_1G_2G_3}{[(1+G_2G_3H_2)+G_2G_3]\frac{H_1}{G_3}} \Rightarrow \frac{G_1G_2\cancel{H_2}G_3}{1+G_2G_3H_2+G_2H_1}$$

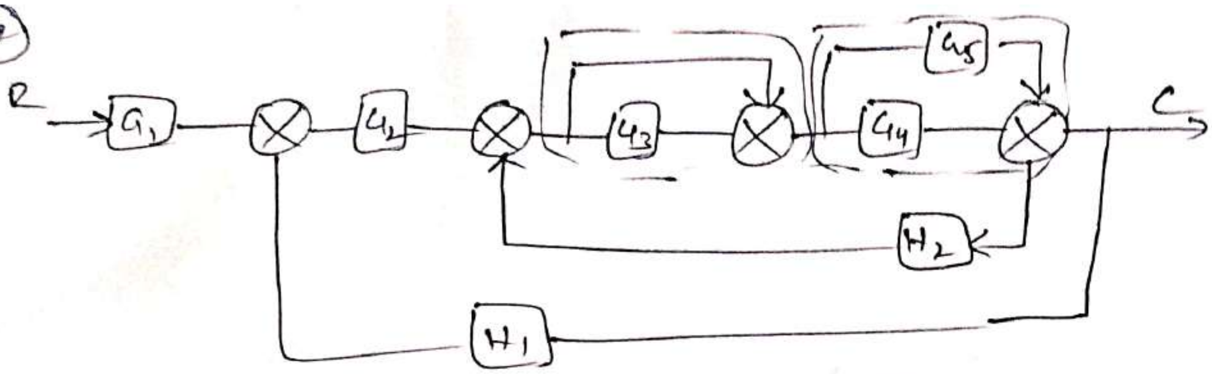


$$\Rightarrow \frac{\frac{G_1G_2G_3}{1+G_2G_3H_2\cancel{+G_1G_2H_1}}}{1+\frac{G_1G_2G_3}{1+G_2G_3H_2\cancel{+G_1G_2H_1}}} \Rightarrow \frac{G_1G_2G_3}{1+G_2G_3H_2\cancel{+G_1G_2H_1}+G_1G_2G_3 \cdot G_1G_2H_1}$$

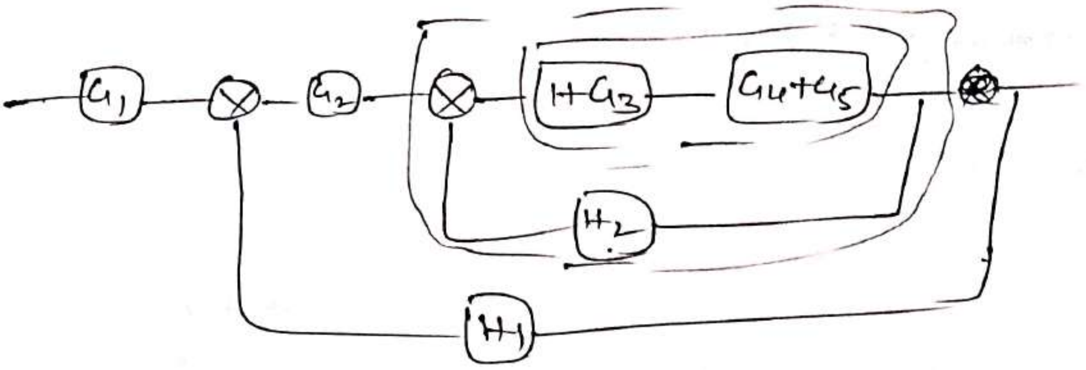
$$\therefore \boxed{\frac{C}{R} = \frac{G_1G_2G_3}{1+G_2G_3H_2+G_1G_2H_1+G_1G_2G_3}}$$

(2)

(2)

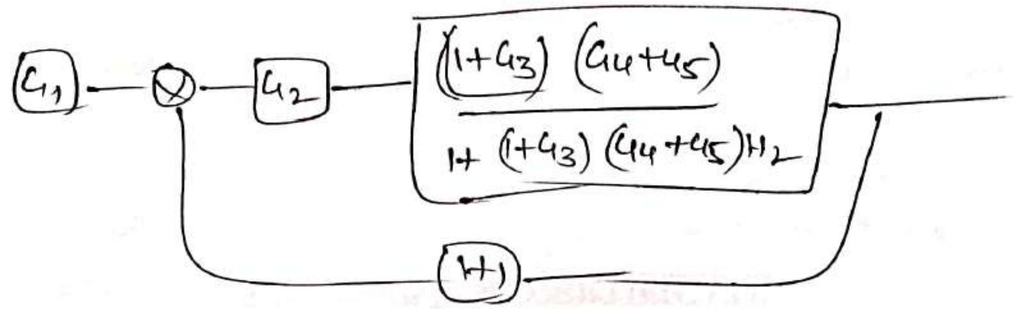


Sol

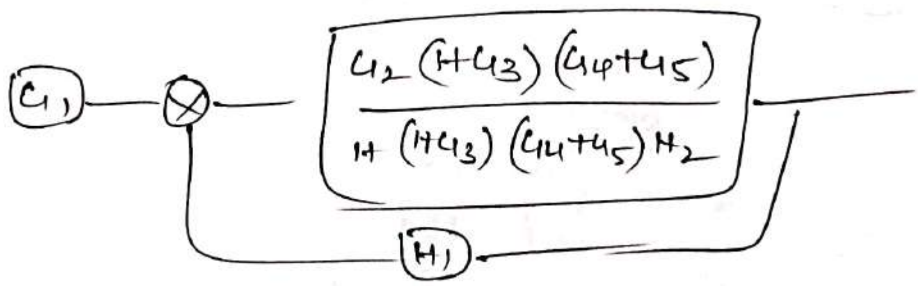


$$\frac{(1+G_3)(G_4+G_5)}{1+}$$

$\Rightarrow$



$\Rightarrow$



$$\Rightarrow \frac{\dfrac{G_2(1+G_3)(G_4+G_5)}{1+(1+G_3)\cancel{(G_4+G_5)H_2}}}{1+\dfrac{G_2(1+G_3)(G_4+G_5)}{\cancel{1+(1+G_3)(G_4+G_5)H_2}}\times H_1} \Rightarrow \frac{G_2(1+G_3)(G_4+G_5)}{1+(1+G_3)(G_4+G_5)H_2+G_2(1+G_3)(G_4+G_5)H_1}$$

$$\Rightarrow \frac{G_1G_2(1+G_3)(G_4+G_5)}{1+(1+G_3)(G_4+G_5)H_2+G_2H_1(1+G_3)(G_4+G_5)} = \frac{C}{R}$$

7/10/12

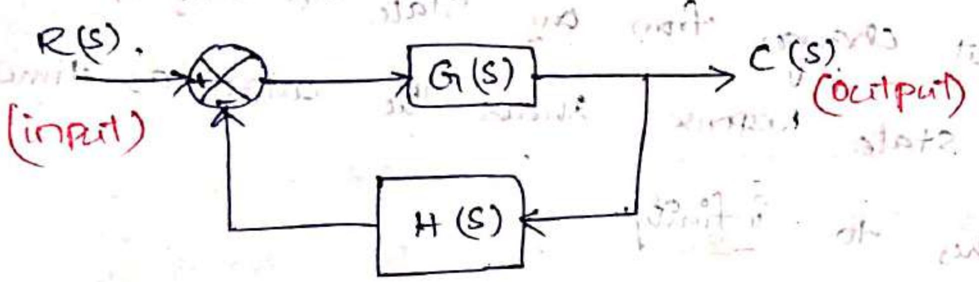
Unit-III

Time Response Analysis

* Time Response :-

The time Response of a System is the output of the closed loop System as a function of time. It is denoted by C(t). The time Response Can be obtained by solving the differential equations governing the System. Alternatively, the Response c(t) Can be obtained from the transfer function of the System and the input to the System.

Consider a closed loop System which is given below,



from the above closed loop transfer function the output in S-domain C(S) is given by the product of the transfer function and its input R(S). Then, on taking the I laplace ~~transfer function~~ transform of this product the time domain Response c(t) is obtained.

from the figure, $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$.

$$\Rightarrow C(S) = R(S)\left[\frac{G(S)}{1+G(S)H(S)}\right] \quad \text{--- (1)}$$

Now Apply I laplace transform function to above eq (1)

$$\Rightarrow L^{-1} C(s) = L^{-1}\left\{R(s)\left[\frac{G_1(s)}{1+G_1(s)H(s)}\right]\right\}$$

$$\Rightarrow \boxed{c(t) = L^{-1}\left[R(s)\,\frac{G_1(s)}{1+G_1(s)H(s)}\right]}.$$

→ The time Response of a Control Systems Consists of two parts :

1. The Transient Response and

2. The steady State Response.

The transient Response shows the output of the System when the input changes from one State to another state. The steady state Response shows the output as time 't' approaches to infinity.

* Standard Test signals :-

The commonly used Test input signals are step, Ramp, impulse, Parabolic and sinusoidal signals.

1. (a) step signal
   (b) unit step signal
2. (a) Ramp signal
   (b) Unit Ramp signal
3. (a) parabolic signal
   (b) unit Parabolic signal.
4. (a) Impulse signal
5. sinusoidal signal.

* Step signal :-

The step signal is a signal whose value changes from Zero to point "A" at $t=0$, then Remains Constant at A for $t>0$. The step signal Resembles an actual steady input to a System. A special Case of step signal is Unit step in which "A" is unity.

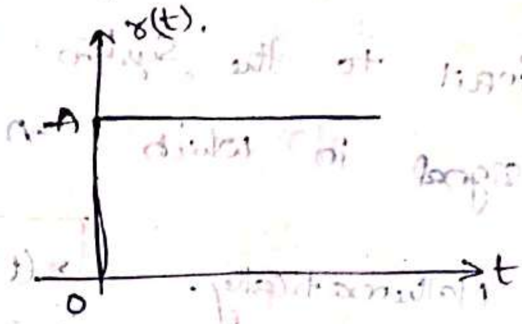
($r(t)$ = Input in time domain)

Mathematically, it is Represented as,

$$\boxed{r(t) = A\,u(t)}$$

where, $u(t) = 1$ for $t \geq 0$.

$u(t) = 0$ for $t < 0$.



* Ramp signal :-

The Ramp signal is a signal whose value increases linearly with time from an initial value of Zero at $t=0$. The Ramp signal Resembles a Constant velocity input to the System.

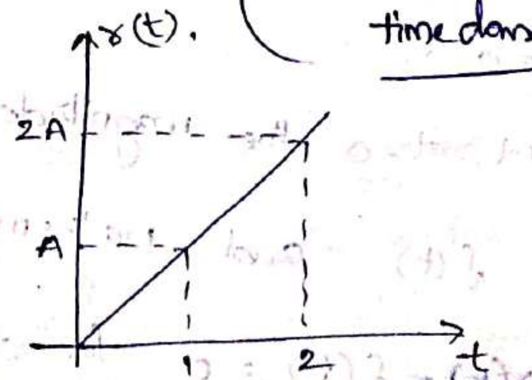
A special Case of Ramp signal is Unit ramp signal in which the value of A is unity.

($r(t)$ = input in time domain)

Mathematically, it is Represented as,

$$\boxed{r(t) = At} \; ; \; t \geq 0$$

$$= 0 \; ; \; t < 0.$$



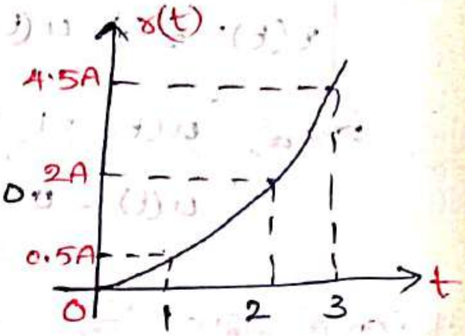
* Parabolic signal :-

The instantaneous value varies a square of the time from an initial value of zero at $t=0$. Now take a look on sketch of the signal with respect to the time resembles a parabola.

The parabolic signal indicates a constant acceleration input to the system. A special case is unit parabolic signal in which A is unity.

Mathematically, $\boxed{r(t) = \frac{At^2}{2}}$; $t \geq 0$.

$= 0$; $t<0$.



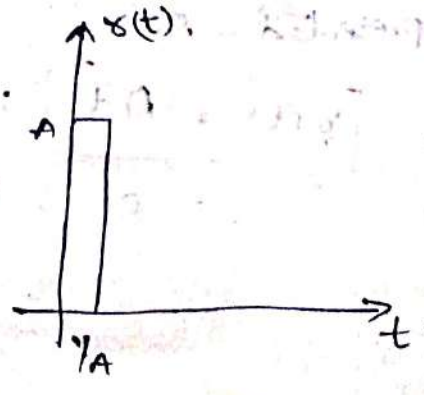
* Impulse signal :-

A signal which is available for very short duration is called impulse signal. Ideal impulse signal is a unit impulse signal which is defined as a signal having zero values at all times except at $t=0$.

At $t=0$ the magnitude becomes infinite. It is denoted by $\delta(t)$ and Mathematically,

$$r(t) = \delta(t) = 0 \quad \text{for } t \neq 0.$$

and $\underset{t_1 \to 0}{Lt} \int_{-t_1}^{+t_1} \delta(t)\, dt = 1$



The time Response of the System $c(t)$ for the impulse signal as input is given by.

$$\boxed{c(t) = L^{-1}\left[R(s)\frac{G(s)}{1+G(s)H(s)}\right] = L^{-1}\left[\frac{G(s)}{1+G(s)H(s)}\right]}.$$

[$\because$ For the Impulse signal $R(s) = 1$]

*** Tabulation of Test signals in both s-domain and in T-domain :-**

Signal	(input in time domain) $r(t)$	(input in s-domain) $R(s)$
Step	A	A/s
Unit step	1	$1/s$
Ramp	At	A/s^2
Unit Ramp	t	$1/s^2$
Parabolic	$At^2/2$	A/s^3
Unit Parabolic	$t^2/2$	$1/s^3$
Impulse	$\delta(t)$	1

$$\left[\because \frac{t^{n-1}}{(n-1)!} \; n=1,2,\ldots \Rightarrow \frac{1}{s^n}\right]$$

*** Order of a System :-**

The input output Relationship of control System can be expressed by a differential equation. The order of a system is given by the order of differential equation governing the System. If the System is governed by n^{th} order differential equation the System is Called n^{th} order of the System.

Now, the transfer function of a System is obtained by taking laplace transform of the differential equation governing the system.

$$\text{Transfer function}, T(S) = K \cdot \frac{P(S)}{Q(S)}$$

where, K = Constant

$P(S)$ = Numerator Polynomial.

$Q(S)$ = Denominator polynomial.

* Time Response of first order system for unit step input :-

The most general form of a differential equation Representing a first order system is expressed as,

$$T \frac{d\,y(t)}{dt} + y(t) = K \cdot x(t).$$

where, $x(t)$, $y(t)$ are input and output variables respectively. and T = Time Constant of a system.

By applying the laplace transform to the above equation we get, $T S\, Y(S) + Y(S) = X(S) \cdot K.$

$$\Rightarrow \boxed{\frac{Y(S)}{X(S)} = \frac{K}{TS+1}}$$

(or).

$$\Rightarrow \boxed{\frac{C(S)}{R(S)} = \frac{K}{TS+1}}$$

$\therefore$ The above transfer function is known as First order transfer function.

for $K=1$

$\Rightarrow$ The first order T.F is $\dfrac{C(S)}{R(S)} = \dfrac{1}{TS+1}$.

For the Unit step signal, the input $R(S) = \dfrac{1}{S}$.

$\therefore$ The Response in S-domain $C(S) = R(S)\,\dfrac{1}{TS+1}$

$$\Rightarrow C(S) = \frac{1}{S(TS+1)}.$$

$$= \frac{1}{ST\left(S+\frac{1}{T}\right)}.$$

$$= \frac{\frac{1}{T}}{S\left(S+\frac{1}{T}\right)}.$$

By partial fractions,

$$\frac{\frac{1}{T}}{S\left(S+\frac{1}{T}\right)} = \frac{A}{S} + \frac{B}{\left(S+\frac{1}{T}\right)}.$$

$$\frac{1/T}{S\left(S+\frac{1}{T}\right)} = \frac{\left(S+\frac{1}{T}\right)A + BS}{\left(S+\frac{1}{T}\right)S}.$$

$$\Rightarrow 1/T = A\left(S+\frac{1}{T}\right) + BS.$$

Now, Substitute $S=0 \Rightarrow A = 1$.

then: Substitute $S = \frac{-1}{T} \Rightarrow B = -1$.

$$\therefore C(S) = \frac{1}{S} - \frac{1}{S+\frac{1}{T}}.$$

Now, inverse laplace transform function,

$$L^{-1} C(S) = L^{-1}\left(\frac{1}{S} - \frac{1}{S+\frac{1}{T}}\right)$$

$$\Rightarrow \therefore \boxed{c(t) = 1 - e^{-t/T}}$$

* Time Response of first order for the Unit Ramp i/p:-

The transfer function of first order system.

$$\frac{C(S)}{R(S)} = \frac{1}{TS+1}.$$

Now for the unit Ramp signal, the input $R(S) = \frac{1}{S^2}$

$$\therefore \quad C(S) = R(S)\frac{1}{1+ST}$$

$$= \frac{1}{S^2(1+ST)}.$$

$$C(S). = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{1+ST}.$$

$$\Rightarrow \frac{1}{S^2(1+ST)} = \frac{A}{S} + \frac{B}{S^2} + \frac{C}{1+ST}$$

$$1 = AS(1+ST) + B(1+ST) + CS^2$$

Now, for $S=0 \Rightarrow A = -T$ (on simplification).

Similarly, $B = 1.$

$C = +T.$

$$\therefore \quad C(S) = \left(\frac{-T}{S} + \frac{1}{S^2} + \frac{T^2}{ST+1}\right)$$

Applying inverse laplace transform we get $\left[L^{-1}\left(\frac{1}{S+a}\right) = e^{-at}\right]$

$$C(t) = L^{-1}\left[\frac{-T}{S} + \frac{1}{S^2} + \frac{T^2}{1+ST}\right]$$

$$\therefore \boxed{C(t) = \left[-T + t + Te^{-t/T}\right].}$$

$$\therefore \boxed{C(t) = -T + t + Te^{-t/T}}$$

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$$\frac{1}{s^2(1+sT)} = \frac{A}{s}+\frac{B}{s^2}+\frac{C}{1+sT}$$

When $R(s)=\frac{1}{s^2}$

$$\Rightarrow \frac{1}{\cancel{s^2(sT+1)}} = \frac{As(1+sT)+B(1+sT)+Cs^2}{\cancel{s^2(1+sT)}}$$

$$\Rightarrow 1 = As(1+sT)+B(1+sT)+Cs^2$$

Sub. $s=0$

$$\Rightarrow \boxed{1 = B}$$

$$s = \frac{-1}{T}$$

$$\Rightarrow 1 = C\left(\frac{-1}{T}\right)^2$$

$$= \frac{C}{T^2} \Rightarrow \boxed{C=T^2}$$

~~Substitute~~ Equating s^2 terms

$$\Rightarrow 0 = AT + C$$

$$\Rightarrow 0 = AT + T^2$$

$$\Rightarrow AT = -T^2$$

$$\Rightarrow \therefore A = -T$$

$$\therefore L^{-1}(C(s)) = L^{-1}\left[\frac{-T}{s}+\frac{1}{s^2}+\frac{T^2}{1+sT}\right]$$

$$c(t) = -T + t + \frac{T^2}{T\left(s+\frac{1}{T}\right)}$$

$$\therefore c(t) = -T + t + Te^{-t/T}$$

$$\because \boxed{\frac{t^{n-1}}{(n-1)!} = \frac{1}{s^n}}$$

$$\Rightarrow \frac{t^{2-1}}{(2-1)!} = \frac{1}{s^2}$$

$$\Rightarrow t = \frac{1}{s^2}$$

* Time Response of first order for the impulse signal i/p :-

first order T.F $= C(s) = R(s)\dfrac{1}{1+sT}$

⇒ for the impulse signal the input $R(s) = 1$

$$\Rightarrow C(s) = \frac{1}{1+sT}$$

$$= \frac{1}{T\left(\frac{1}{T}+s\right)}$$

⇒ Now, Apply inverse laplace transform to above equation,

$$\left[\because L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}\right]$$

$$c(t) = L^{-1}\left[\frac{1}{T\left(s+\frac{1}{T}\right)}\right]$$

$$= \frac{1}{T}L^{-1}\left(\frac{1}{s+\frac{1}{T}}\right)$$

$$= \frac{1}{T}e^{-t/T}$$

$$\therefore \boxed{c(t) = \frac{1}{T}e^{-t/T}}$$

* Second order System :-

The standard form of closed loop transfer function for second order system is given by.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2s\,\omega_n\zeta+\omega_n^2}$$

where, ω_n = Natural frequency, rad/sec

ζ = Damping Ratio.

The Damping Ratio is defined as the ratio of actual Damping to the critical damping.

Depending up on the value of ξ, the system can be classified into following four cases.

Case-i: Undamped system, $\xi = 0$.

Case-ii: under damped system, $0 < \xi < 1$.

Case-iii: critically damped system, $\xi = 1$

Case-iv: over damped system, $\xi > 1$.

* Characteristic equation of a system :-

The closed loop transfer function of a system is given by,

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}.$$

Now the equation which gives the poles of a system is defined as characteristic equation, $\Rightarrow 1+G(S)H(S)=0$

for first order system, $\frac{C(S)}{R(S)} = \frac{1}{1+ST}$

$\Rightarrow S = \frac{-1}{T}$ Root of first order system

Similarly, for 2nd order system,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+2S\omega_n\xi+\omega_n^2}$$

The C.E $\Rightarrow 1+G(S)H(S) = 0$

$\Rightarrow S^2+2S\omega_n\xi+\omega_n^2 = 0$

Since it is a quadratic equation, therefore the roots are given by, $\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2-1)}}{2}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

* Response of undamped system for the unit step input :-

Standard 2nd order transfer function is,

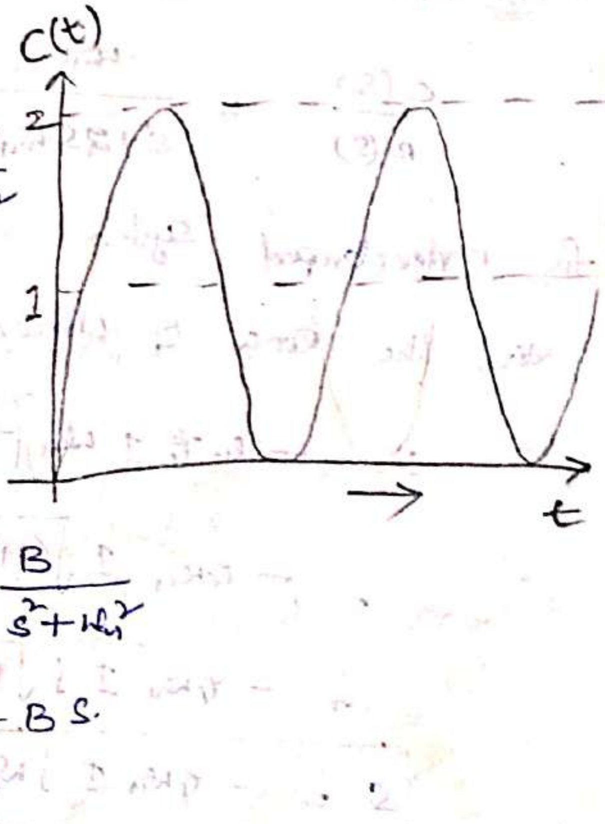
$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{2\zeta\omega_n S+\omega_n^2+S^2}$$

for undamped system $\zeta = 0$.

$$\therefore \frac{C(S)}{R(S)} = \frac{\omega_n^2}{S^2+\omega_n^2}$$

Now, we have to find the Response for unit step.

$r(t) = 1 \Rightarrow R(S) = \frac{1}{S}$



$$\Rightarrow C(S) = R(S)\frac{\omega_n^2}{S^2+\omega_n^2}$$

$$= \frac{\omega_n^2}{S(S^2+\omega_n^2)}$$

$\Rightarrow$ By partial fractions,

$$\frac{\omega_n^2}{S(S^2+\omega_n^2)} = \frac{A}{S} + \frac{B}{S^2+\omega_n^2}$$

$$\Rightarrow \omega_n^2 = A(S^2+\omega_n^2) + BS.$$

Now, substitute $S = 0 \Rightarrow A = 1.$

Similarly, Substitute $s = j\omega_n$.

$$\omega_n^2 = A(-\omega_n^2 + \omega_n^2) + B j\omega_n$$

$$\Rightarrow B = \frac{\omega_n^2}{j\omega_n}$$

$$= -j\omega_n = -s.$$

Note :- Every practical system has some amount of damping. Hence undamped system does not exist in practice

$\therefore A = 1, \quad B = -s.$

$$\therefore C(s) = \frac{1}{s} - \frac{s}{\omega_n^2 + s^2}$$

$$\left[\because L^{-1}\frac{s}{s^2+\omega_n^2} = \cos\omega_n t \qquad L^{-1}\frac{\omega_n}{s^2+\omega_n^2} = \sin\omega_n t\right]$$

Now, $$c(t) = L^{-1}\left[\frac{1}{s} - \frac{s}{s^2+\omega_n^2}\right]$$

$$\therefore \boxed{c(t) = 1 - \cos\omega_n t}$$

* Response of second order system for underdamped when input is unit step :-

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2s\omega_n\zeta + \omega_n^2}$$

for underdamped system, $0 < \zeta < 1$

Now the Roots of characteristic equation,

$$s = -\omega_n\zeta \pm \omega_n\sqrt{[-1+\zeta^2]}$$

$$= -\zeta\omega_n \pm \sqrt{(-1)(1-\zeta^2)}\,\omega_n$$

$$= -\zeta\omega_n \pm j\sqrt{1-\zeta^2}\,\omega_n$$

$$\boxed{s = -\zeta\omega_n \pm j\omega_d.} \qquad \left[\because \omega_d = \omega_n\sqrt{1-\zeta^2}\right]$$

where, ω_d = Damped frequency of oscillation.

Response in 's' domain $c(s) = R(s) \dfrac{\omega_n^2}{s^2+2s\omega_n\zeta+\omega_n^2}$

for unit step input $R(s) = 1/s$.

$$\Rightarrow c(s) = \frac{1}{s} \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

By taking partial fractions,

$$\frac{1 \ \omega_n^2}{s\left(s^2+2\omega_n\zeta s+\omega_n^2\right)} = \frac{A}{s} + \frac{Bs+C}{\left(s^2+2\omega_n s\cdot\zeta+\omega_n^2\right)}.$$

$$\Rightarrow \omega_n^2 = A\left(s^2+2\omega_n\zeta s+\omega_n^2\right)+(Bs+C)s.$$

$$\Rightarrow \omega_n^2 = A\left(s^2+2\omega_n\zeta s+\omega_n^2\right) + Bs^2+Cs.$$

Equating the coefficients of 's^2' $\Rightarrow 0 = 1+B$ $\left[\because A=1\right]$

$$\Rightarrow B = -1.$$

and equating the 's' coefficients $\Rightarrow 0 = 2\zeta\omega_n + C$

$$\Rightarrow C = -2\zeta\omega_n.$$

$\therefore$ Substitute $s=0 \Rightarrow A=1.$

$$\therefore c(s) = \frac{1}{s} + \frac{(Bs+C)}{s^2+2\omega_n\zeta s+\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+2\omega_n\zeta)}{s^2+2\omega_n\zeta s+\omega_n^2}.$$

Now, adding and substracting $\zeta^2\omega_n^2$ to denominator for the second term,

$$\Rightarrow c(s) = \frac{1}{s} - \frac{(s+2\omega_n\zeta)}{s^2+2\omega_n\zeta s+\omega_n^2+\zeta^2\omega_n^2-\zeta^2\omega_n^2}$$

$$\Rightarrow c(s) = \frac{1}{s} - \frac{(s+2\omega_n\xi)}{s^2+(2\omega_n\xi s+\xi^2\omega_n^2)+\omega_n^2-\xi^2\omega_n^2}$$

$$= \frac{1}{s} - \frac{(s+2\omega_n\xi)}{s^2+2\omega_n\xi s+\xi^2\omega_n^2+\omega_n^2(1-\xi^2)}$$

$$= \frac{1}{s} - \frac{(s+2\omega_n\xi)}{s^2+2\omega_n\xi s+\xi^2\omega_n^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{(s+2\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2}$$

$$= \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\omega_n\xi)^2+\omega_d^2} - \frac{\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}$$

Now, Multiplying and dividing ω_d in third term of the denominator we get,

$$c(s) = \frac{1}{s} - \frac{(s+\xi\omega_n)}{(s+\xi\omega_n)^2+\omega_d^2} - \frac{\xi\omega_n}{\omega_d}\cdot\frac{\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

Now, apply the inverse laplace transform function,

$$\Rightarrow c(t) = 1 - e^{-\xi\omega_n t}\cos\omega_d t - \frac{\xi\omega_n}{\omega_d}e^{-\xi\omega_n t}\sin\omega_d t$$

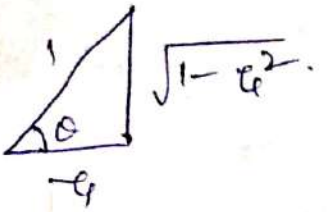
$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{-\xi\cancel{\omega_n}}{\cancel{\omega_n}\sqrt{1-\xi^2}}\sin\omega_d t\right].$$

$$= 1 - e^{-\xi\omega_n t}\left[\cos\omega_d t + \frac{-\xi}{\sqrt{1-\xi^2}}\sin\omega_d t\right]$$

$$= 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left(\sqrt{1-\xi^2}\cos\omega_d t + -\xi\sin\omega_d t\right).$$

$$\left[\because L^{-1}\left[\frac{\omega}{(s+a)^2+\omega^2}\right] = e^{-at}\sin\omega t\right] \qquad \left[L^{-1}\left[\frac{(s+a)}{(s+a)^2+\omega^2}\right] = e^{-at}\cos\omega t\right]$$

If we construct a right angled triangle with the help of $\sqrt{1-\zeta^2}$ and ζ, we get,



$$\sin\theta = \sqrt{1-\zeta^2}$$
$$\cos\theta = \zeta.$$
$$\tan\theta = \frac{\sqrt{1-\zeta^2}}{\zeta}.$$

Now,

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin\theta \cos\omega_d t + \cos\theta \sin\omega_d t\right]$$

$$\therefore \boxed{c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta).}$$

* Response of second order system for critically damped system for step (unit) input :-

Note:- Critically damped system does not have any oscillations.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2s\omega_n\zeta+\omega_n^2}$$

for critically damped system $\zeta = 1$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+\omega_n^2+2\omega_n\zeta s} = \frac{\omega_n^2}{s^2+2\omega_n s+\omega_n^2}$$

for unit step input $R(s) = 1/s$.

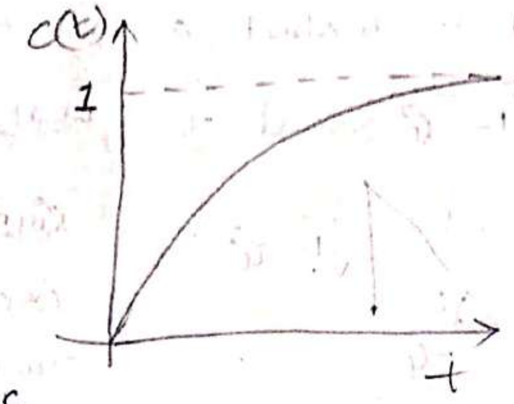
$$\Rightarrow C(s) = \frac{1}{s}\,\frac{\omega_n^2}{s^2+\omega_n^2+2\omega_n s} \qquad [\because (a+b)^2 = a^2+b^2+2ab]$$

$$= \frac{\omega_n^2}{s(s+\omega_n)^2}.$$

By partial fractions,

$$\frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n}.$$

* on symplifying.



$A = 1.$

$B = -\omega_n.$

$C = -1.$

$$\therefore C(S) = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{s+\omega_n},$$

$$= \frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} + \frac{(-1)}{s+\omega_n}.$$

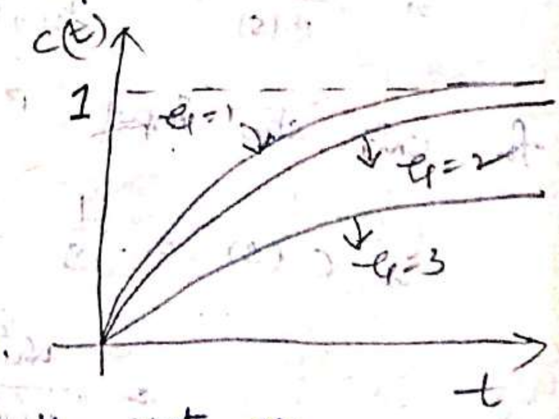
$$\therefore c(t) = L^{-1}\left[\frac{1}{s} - \frac{\omega_n}{(s+\omega_n)^2} - \frac{1}{s+\omega_n}\right]. \quad \left[\because L^{-1}\left(\frac{1a}{(s+a)^2}\right) = at\,e^{-at}\right]$$

$$= 1 - \omega_n t\, e^{-\omega_n t} - e^{-\omega_n t}.$$

$$\therefore \boxed{c(t) = 1 - e^{-\omega_n t}(\omega_n t + 1)}$$

* Response of second order system for over damped condition when input is unit step signal :-

The standard form of closed loop transfer function of second order system is,



$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\omega_n s\xi+\omega_n^2}$$

for over damped system $\xi > 1$.

from the characteristic equation the roots are

$$s_1, s_2 = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

$$= -\left[\xi\omega_n \mp \omega_n\sqrt{\xi^2-1}\right].$$

Now, assume, $s_a = -\xi\omega_n + \omega_n\sqrt{\xi^2-1}$

$s_b = -\xi\omega_n - \omega_n\sqrt{\xi^2-1}$

Now, let $s_1 = -s_a$ and $s_2 = -s_b$.

Now, Closed loop transfer function can be written interms of s_1 and s_2 is given by,

$$\frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2+2\omega_n s\xi+\omega_n^2}$$

$$= \frac{\omega_n^2}{(s+s_1)(s+s_2)}.$$

Note :- Over damped system does not have any oscillations. But it takes longer time for Response to reach its final steady value

That means, to produce the desired o/p it takes long time (or) more time.

for unit step signal $R(s) = 1/s$.

$$\Rightarrow C(S) = \frac{R(S).\ \omega_n^2}{(s+s_1)(s+s_2)}$$

$$= \frac{\omega_n^2}{s(s+s_1)(s+s_2)}$$

By Partial fractions,

$$\frac{\omega_n^2}{s(s+s_1)(s+s_2)} = \frac{A}{s} + \frac{B}{s+s_1} + \frac{C}{s+s_2}$$

for $s = 0$.

$$\Rightarrow \omega_n^2 = A(s+s_1)(s+s_2) + B\,s(s+s_2) + C\,s(s+s_1).$$

$$\therefore \omega_n^2 = A\left(-\xi\omega_n - \omega_n\sqrt{\xi^2-1}\right)\left(-\xi\omega_n + \omega_n\sqrt{\xi^2-1}\right)$$

$$\Rightarrow \omega_n^2 = A\left[\xi^2\omega_n^2 - \cancel{\xi\omega_n^2\sqrt{\xi^2-1}} + \cancel{\omega_n^2\xi\sqrt{\xi^2-1}} - \omega_n^2(\xi^2-1)\right]$$

$$\Rightarrow \cancel{\omega_n^2} = A\,\cancel{\omega_n^2} \Rightarrow A = 1$$

$$\Rightarrow A = 1.$$

Similarly,

for $s = -s_1$ $\Rightarrow$ $B = \frac{-\omega_n}{2\sqrt{\xi^2-1}} \cdot \frac{1}{s_1}.$

for $s = -s_2 \Rightarrow C = \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_2}$

$$\therefore C(s) = \frac{1}{s} - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_1} \cdot \frac{1}{s+s_1} + \frac{\omega_n}{2\sqrt{\zeta^2-1}} \cdot \frac{1}{s_2} \cdot \frac{1}{s+s_2}$$

Now, on taking inverse laplace transform we get,

$$c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \frac{1}{s_1} e^{-s_1 t} + \frac{\omega_n}{2\sqrt{\zeta^2-1}} \frac{1}{s_2} e^{-s_2 t}$$

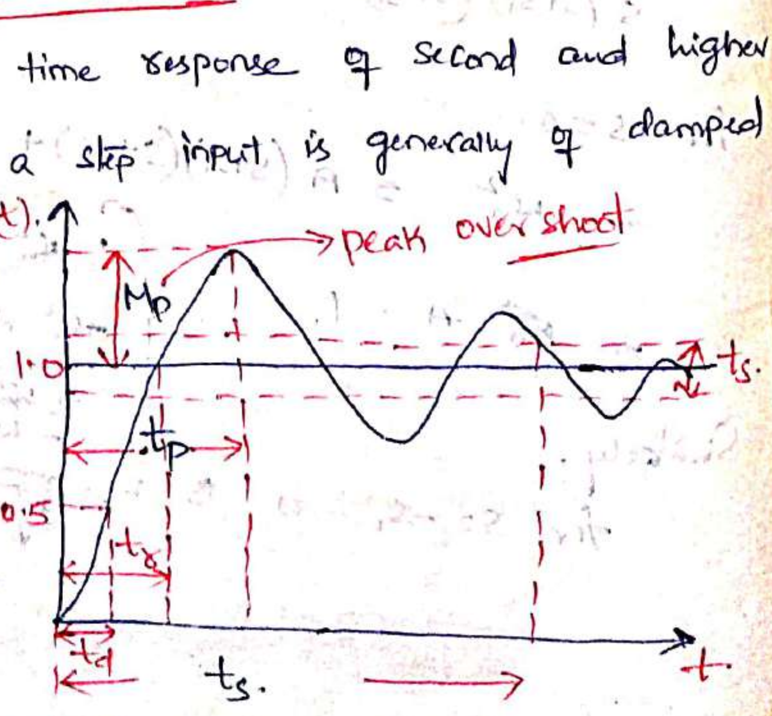
$$\boxed{\therefore c(t) = 1 - \frac{\omega_n}{2\sqrt{\zeta^2-1}} \left(\frac{e^{-s_1 t}}{s_1} - \frac{e^{-s_2 t}}{s_2} \right).}$$

where, $s_1 = -\left[-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right]$

$s_2 = -\left[\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right]$

* Time Domain Specifications :-

Control Systems are generally designed with damping less than one, i.e., oscillatory step response. High order control systems usually have a pair of complex conjugate poles with damping less than one which dominate over the other poles. Therefore, the time response of second and higher order control systems to a step input is generally of damped oscillatory nature, which is shown below.



The desired performance characteristics of any control system are specified in terms of Time domain specifications.

The transient Response characteristics of a Control System to a unit step is specified interms of following time domain specifications.

1. Delay time (t_d).
2. Rise time (t_r).
3. peak time. (t_p)
4. Maximum over shoot (M_p).
5. Settling time (t_s).

∵ Control Systems are generally designed with $\zeta<1$ i.e., under damped case ⇒ $0<\zeta<1$

The desired Performance characteristics of any System of any Order may be specified in terms of transient response to a unit step i/p signal.

1. Delay time :-

It is the time taken for response to reach 50% of the final value, for the very first time.

2. Rise time :-

It is the time taken for Response to raise from 0 to 100% for very first time. For underdamped System the rise time is calculated from 0% to 100%. But for overdamped System it is the time taken by the response to raise from 10% to 90%. For critically damped Systems, it is the time taken for response to raise from 5% to 95%.

3. peak time :-

It is the time taken for the Response to reach the peak value for the very first time. (or) It is the time taken for the Response to reach peak over shoot (M_p).

4. Peak overshoot :-

Is defined as the ratio of Maximum peak value measured from ~~final~~ Maximum value to the final value.
final,

let final value $= c(\infty)$.

Maximum value $= c(t_p)$.

$$\text{peak overshoot}, M_p = \frac{c(t_p) - c(\infty)}{c(\infty)}$$

$$\boxed{\% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100}$$

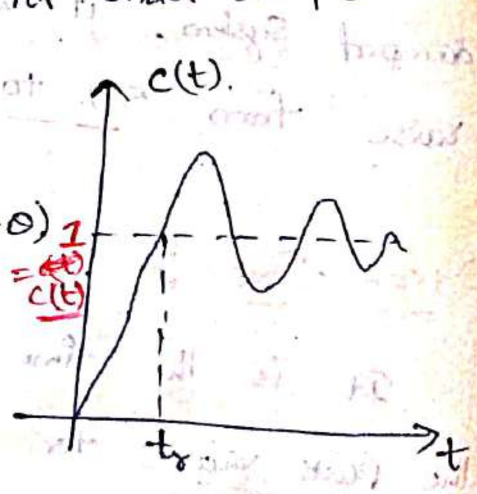
5. Settling time :-

It is defined as the time taken by the Response to reach and stay with in a specified error. It is commonly expressed as % of final value. The usual tolerance error is 2% (or) 5% of the final value

* Expressions for the Time Domain Specifications :-

① Rise time (t_r) :-

Response of second order system for under damped case is given by,



$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

At $t = t_r$, $c(t) = c(t_r) = 1$.

Since from the Time Response figure at $t = t_r$ the output $c(t) = 1$.

$\therefore c(t_r) = 1 - \frac{e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta) = 1$

$\Rightarrow \therefore \frac{-e^{-\zeta\omega_n t_r}}{\sqrt{1-\zeta^2}} \left(\sin(\omega_d t_r + \theta)\right) = 0.$

since we have assumed the under damped system. For this $\zeta = 0 < \zeta < 1.$

$\therefore -e^{-\zeta\omega_n t_r} \neq 0.$ and $\sin(\omega_d t_r + \theta) = 0$

$\Rightarrow \sin(\omega_d t_r + \theta) = 0$

$\Rightarrow \omega_d t_r + \theta = \pi$

$\Rightarrow t_r = \frac{\pi - \theta}{\omega_d}.$

$\left[\because \sin(\omega_d t_r + \theta) = 0 \Rightarrow \sin(\phi) = 0. \Rightarrow \phi = 0, \pi, 2\pi, 3\pi, \ldots \right]$

$\therefore \boxed{t_r = \frac{\pi - \theta}{\omega_d}.}$

Here, $\boxed{\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right).}$

$\therefore$ Here we are calculating for half cycle which is π. If we calculate for full cycle 2π which is equal to $0° : 360° = 0°$.

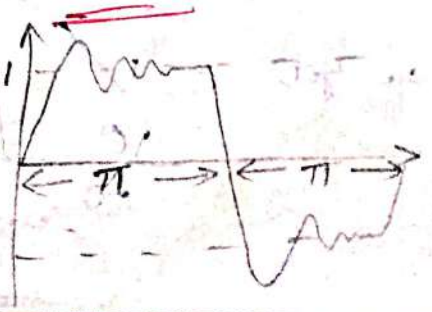
and ω_d = Damped frequency of oscillation.

$\therefore \boxed{\omega_d = \omega_n\sqrt{1-\zeta^2}}$

$\therefore$ Rise time $\boxed{t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}}{\omega_n\sqrt{1-\zeta^2}} \text{ in secs.}}$

the angle $\theta = \tan^{-1}\left\{\frac{\sqrt{1-\zeta^2}}{\zeta}\right.$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$ in radians.



② Peak time (t_p) :-

To find the expression for peak time, t_p, differentiate $c(t)$ with respect to t and equate to zero.

i.e, $$\frac{d}{dt} c(t)\Big|_{t=t_p} = 0.$$

The unit step response of second order system for the underdamped condition is given by, $\left[\frac{d}{dt} e^{-ax} = -a e^{-ax}\right]$

$$c(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta).$$

Now, differentiate $c(t)$ with respect to t we get,

$$\frac{d}{dt} c(t) = \frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (-\zeta\omega_n)\left(\sin(\omega_d t + \theta)\right) + \left(\frac{-e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \cos(\omega_d t + \theta) \times \omega_d\right]$$

Now, put $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

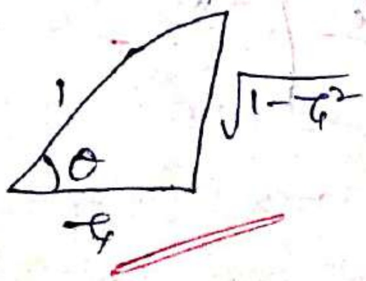
$$\frac{d}{dt} c(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} (\zeta\omega_n) \sin(\omega_d t + \theta) - \frac{\omega_n\sqrt{1-\zeta^2}}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t + \theta)$$

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \left[\zeta \sin(\omega_d t + \theta) - \sqrt{1-\zeta^2} \cos(\omega_d t + \theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \left[\cos\theta \sin(\omega_d t + \theta) - \sin\theta \cos(\omega_d t + \theta)\right]$$

[∵ By constructing Right angle triangle with ζ & $\sqrt{1-\zeta^2}$

we get,



⇒ $\sin\theta = \sqrt{1-\zeta^2}$

$\cos\theta = \zeta$.]

$$\therefore \frac{d\,c(t)}{dt} = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\sin(\omega_d t+\theta)\cos\theta - \cos(\omega_d t+\theta)\sin\theta\right].$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t}\left[\sin((\omega_d t+\theta)-\theta)\right]$$

$$= \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t)$$

Now, Substitute $t = t_p$ & equate $\frac{d\,c(t)}{dt} = 0$.

$$\Rightarrow \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \sin(\omega_d t_p) = 0.$$

[$\because \sin(\omega_d t_p) = \pi$ అపుడే total term zero అవుతుంది]

Since, for underdamped system the damping ratio ζ is $0<\zeta<1$. So, the term $\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t_p} \neq 0$.

and $\sin(\omega_d t_p) = 0$.

$$\Rightarrow \omega_d t_p = \pi$$

$$\therefore t_p = \frac{\pi}{\omega_d}.$$

[$\sin(\omega_d t_p) = 0$
$\Rightarrow \sin\phi = 0.$
when $\phi = 0, \pi, 2\pi, 3\pi, \ldots$]

ω_d = Damped frequency of oscillation.

$$\therefore \text{The peak time } t_p = \frac{\pi}{\omega_d}$$

$$= \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$\therefore \boxed{t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}}$$

$\rightarrow$ peak time for the second order system at the condition of underdamped system.

③ peak overshoot (M_p) :-

$$\% \text{ peak overshoot } \% M_p = \frac{c(t_p) - c(\infty)}{c(\infty)} \times 100\%.$$

where, $c(t_p)$ = peak value at $t = t_p$.

$c(\infty)$ = Final steady state value.

The unit step response of second order system is given by,

$$c(t) = 1 - \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

Now, At $t = \infty$

$$\Rightarrow c(t) = c(\infty) = 1 - \frac{e^{-\infty}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \theta)$$

$$= 1 - 0 = 1.$$

$$\left[\because e^{-\infty} = \frac{1}{e^{+\infty}} = \frac{1}{\infty} = 0 \right]$$

At $t = t_p$.

$$c(t_p) = 1 - \frac{e^{-\zeta \omega_n t_p}}{\sqrt{1-\zeta^2}} \sin(\omega_d t_p + \theta)$$

Now, put, $t_p = \frac{\pi}{\omega_d}$.

$$\Rightarrow c(t_p) = 1 - \frac{e^{-\zeta \omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\zeta^2}} \sin\left(\cancel{\omega_d} \cdot \frac{\pi}{\cancel{\omega_d}} + \theta\right)$$

then substitute $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$.

$$\therefore c(t_p) = 1 - \frac{e^{-\zeta \cancel{\omega_n} \frac{\pi}{\cancel{\omega_n}\sqrt{1-\zeta^2}}}}{\sqrt{1-\zeta^2}} \sin(\pi + \theta).$$

$$= 1 + \frac{e^{-\zeta\pi/\sqrt{1-\zeta^2}}}{\sqrt{1-\zeta^2}} \sin\theta.$$

$[\because \sin(\pi + \theta) = -\sin\theta]$

from the Right angled triangle,

$$\sin\theta = \sqrt{1-\xi^2}$$

$$\therefore C(t_p) = 1 + \frac{e^{-\xi\pi/\sqrt{1-\xi^2}}}{\cancel{\sqrt{1-\xi^2}}} \times \cancel{\sqrt{1-\xi^2}}$$

$$\therefore \boxed{C(t_p) = 1 + e^{-\xi\pi/\sqrt{1-\xi^2}}}$$

Now, peak overshoot $\Rightarrow$ $\% M_p = \frac{C(t_p) - C(\infty)}{C(\infty)} \times 100\%$

$$\Rightarrow \% M_p = \frac{\cancel{1} + e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} - \cancel{1}}{1} \times 100\%$$

$$\Rightarrow \therefore \boxed{\% M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}} \times 100\%}$$

④ Settling time :-

The Response of the second order system has two components. They are

1. Decaying exponential component, $\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}$
2. sinusoidal component, $\sin(\omega_d t + \theta)$.

Here, the exponential component is used to Reduce the oscillations produced by sinusoidal component.

The settling time is determined by the exponential component. This is found by equating exponential component to the percentage of tolerance errors.

for 2% tolerance at $t = t_s$, $\dfrac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} = 2\%$

$$\Rightarrow \frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} = 0.02$$

for $\sqrt{1-\zeta^2} \ll 1$ $\Rightarrow$ $e^{-\zeta \omega_n t} = 0.02$

(or) for least values of ζ, $e^{-\zeta \omega_n t} = 0.02$

$$\Rightarrow -\zeta \omega_n t_s = \ln(0.02)$$

$$\Rightarrow -\zeta \omega_n t_s = -4.$$

$$\Rightarrow \boxed{t_s = \frac{4}{\zeta \omega_n}}$$

Similarly, for 5% tolerance,

$$\frac{e^{-\zeta \omega_n t}}{\sqrt{1-\zeta^2}} = 5\%$$

for least values of ζ, $\Rightarrow e^{-\zeta \omega_n t} = 0.05.$

$$\Rightarrow -\zeta \omega_n t_s = \ln(0.05)$$

$$\Rightarrow -\zeta \omega_n t_s = -3.$$

$$\Rightarrow \boxed{t_s = \frac{3}{\omega_n \zeta}}$$

$\therefore$ For 2% tolerance, $t_s = \dfrac{4}{\omega_n \zeta}$

For 5% tolerance, $t_s = \dfrac{3}{\omega_n \zeta}$

In general for a specified percentage error setting time is $\boxed{t_s = \dfrac{\ln(\%.\text{error})}{-\zeta \omega_n}}$

* Type number of control systems :-

The type number is specified for open loop transfer function G(S) H(S). The number of poles of the loop transfer function lying at the origin is decides the type number of the system.

In general, the loop transfer function can be expressed as a ratio of two polynomials, such as Numerator polynomial and Denominator polynomial in S,

$$G(S)H(S) = K \frac{P(S)}{Q(S)}$$

$$= K \cdot \frac{(S+Z_1)(S+Z_2)(S+Z_3)\cdots}{S^N (S+P_1)(S+P_2)(S+P_3)+\cdots}$$

where, Z_1, Z_2, Z_3 are the zeros of the transfer function

P_1, P_2, P_3 are the poles of the transfer function.

K = Constant and

N = Number of poles at the origin.

Simply, the value of N decides the TYPE Number of a Control System

If $N=0$, then it is type-0 system

If $N=1$, " " type-1 system

If $N=2$, " " type-2 system and so on.

* Steady State Response :-

Definition :- The steady state Response of a system is defined as the response of the system as time "t" tends to infinity ($t \to \infty$).

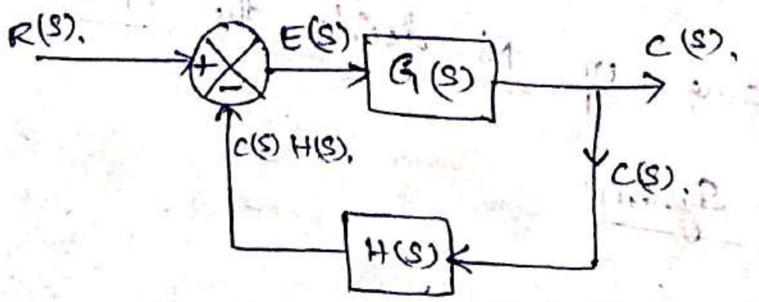
* Steady State Error :-

Steady state error is an extremely important aspect of system behaviour. The steady state error (e_{ss}) is the difference between the input (or desired value) and the output of a closed loop system for a known input as $t \to \infty$.

$$\lim_{t\to\infty}\big(c(t)-r(t)\big) = \lim_{t\to\infty} e(t).$$

The steady state error is a measure of system accuracy. These errors arise from the nature of ~~performance~~ ~~of~~ ~~a~~ inputs, type of system and from non-linearity of system components.

Now, consider a closed loop system,



[Negative feedback System]

where, $E(S)$ = Error signal,

$C(S)H(S)$ = feedback signal

$C(S)$ = output signal.

The Error signal, $E(S) = R(S) - C(S)H(S)$. —— ①

The output signal, $C(S) = E(S)G(S)$. —— ②

Substitute ② in ①

$\Rightarrow E(S) = R(S) - E(S)G(S)H(S)$.

$\Rightarrow E(S) + E(S)G(S)H(S) = R(S)$.

$\Rightarrow E(S)\left[1 + G(S)H(S)\right] = R(S)$.

$$\Rightarrow \boxed{E(S) = \frac{R(S)}{1+G(S)H(S)}}$$ → for Negative feedback System

This is Error signal in s-domain. To obtain in T-domain take inverse laplace transform function of $E(S)$.

$$\Rightarrow L^{-1}\left[E(S)\right] = L^{-1}\left[\frac{R(S)}{1+G(S)H(S)}\right]$$

$$\Rightarrow \boxed{e(t) = L^{-1}\left[\frac{R(S)}{1+G(S)H(S)}\right]}$$

Now, the steady state error (e_{ss}) is defined as the $e(t)$ when $t \to \infty$.

$$\Rightarrow e_{ss} = \underset{t \to \infty}{Lt}\ e(t).$$

∴ for positive F.B System

$$E(S) = \frac{R(S)}{1-G(S)H(S)}$$

From the final value theorem,

If $F(S) = L\{f(t)\}$ then, $\underset{t \to \infty}{Lt} f(t) = \underset{S \to 0}{Lt}\ S\,F(S)$.

Similarly, steady state error

$$e_{ss} = \underset{t\to\infty}{Lt}\ e(t) = \underset{s\to 0}{Lt}\ s\,E(s).$$

$$= \underset{s\to 0}{Lt}\ s\left[\frac{R(s)}{1+G(s)H(s)}\right]$$

$$\therefore \boxed{e_{ss} = \underset{s\to 0}{Lt}\ \frac{s\,R(s)}{1+G(s)H(s)}}$$

* Static error Constants :-

When a Control system is excited with standard input signal then steady state error may zero, Constant or infinite. The value of steady state error depends on the type number and input signal applying to the system.

→ Type-0 System will have a Constant steady state error when the input is step signal. This Constant is known as positional Error Constant

→ Type-1 System will have a Constant steady state error when the input is Ramp signal. And Constant is known as Velocity error Constant

→ Type-2 System will have a Constant steady state error when the input is parabolic signal. And Constant is named as Acceleration error Constant

And the above 3 errors are expressed as, Positional error Constant, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s)$.

Velocity error Constant, $K_v = \underset{s\to 0}{Lt}\ s \cdot G(s)H(s)$.

Acceleration error Constant, $K_a = \underset{s\to 0}{Lt}\ s^2 \cdot G(s)H(s)$.

$\therefore K_p, K_v, K_a$ are generally called as static error Constants.

* Steady state error when the input is unit step signal :-

Generally, steady state error $(e_{ss}) = \underset{s\to 0}{Lt} \dfrac{s\,R(s)}{G(s)H(s)+1}$

when the input is unit step, $R(s) = 1/s$.

$$\Rightarrow \therefore e_{ss} = \underset{s\to 0}{Lt} \frac{\cancel{s}\times\frac{1}{\cancel{s}}}{1+G(s)H(s)}$$

$$= \frac{1}{1+\underset{s\to 0}{Lt}\ G(s)H(s)}$$

$$= \frac{1}{1+K_p}.$$

where, $K_p = \underset{s\to 0}{Lt}\ G(s)H(s)$ is positional error constant

for type-0 System :-

$$K_p = \underset{s\to 0}{Lt}\ G(s)H(s) = \underset{s\to 0}{Lt}\ K \frac{(s+z_1)(s+z_2)(s+z_3)\cdots}{(s+P_1)(s+P_2)(s+P_3)\cdots}$$

$$= \underset{s\to 0}{Lt}\ K \frac{z_1 z_2 z_3}{P_1 P_2 P_3}$$

$$= \text{Constant}$$

$$\therefore \boxed{e_{ss} = \frac{1}{1+K_p}} = \frac{1}{1+\text{Constant}} = \text{Constant}$$

for type-1 system :-

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{s(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= K \frac{(s+z_1)(s+z_2)(s+z_3)}{0}$$

$$= \infty.$$

$\therefore\ K_p = \infty.$

$$\Rightarrow\ e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = \frac{1}{\infty} = 0.$$

for type-2 system :-

$$K_p = \underset{s\to 0}{Lt}\, G(s)H(s) = \underset{s\to 0}{Lt}\, K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{s^2(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= K \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{0}$$

$$= \infty.$$

$$\therefore\ e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0.$$

Note:- for only type-0 system the K_p = Constant there by steady state error e_{ss} = Constant. from type-1 and above higher orders the $K_p = \infty$, Consequently the $e_{ss} = 0$.

* steady state error when the input is Ramp signal :-

Generally, steady state error $e_{ss} = \underset{s\to 0}{Lt} \dfrac{s^1 R(s)}{1+G(s)H(s)}$,

for unit Ramp signal $R(s) = 1/s^2$

$$\Rightarrow \underset{s\to 0}{Lt} \frac{\cancel{s}\frac{1}{s^{\cancel{2}}}}{1+G(s)H(s)} = \underset{s\to 0}{Lt} \frac{1/s}{1+G(s)H(s)}.$$

$$e_{ss} = \underset{s\to 0}{Lt} \frac{1}{s + s\,G(s)H(s)}$$

$$= \frac{1}{0 + \underset{s\to 0}{Lt}\, s\,G(s)H(s)}$$

$$\therefore \boxed{e_{ss} = \frac{1}{K_V}}$$

where, $K_V = \underset{s\to 0}{Lt}\; s\,G(s)H(s)$ is known as velocity error

Constants.

for type-0 System :-

$$K_V = \underset{s\to 0}{Lt}\left[s\,G(s)H(s)\right] = \frac{s\,(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$\Rightarrow K_V = 0$$

$$\therefore e_{ss} = \frac{1}{K_V} = \frac{1}{0} = \infty.$$

for type-1 System :-

$$K_V = \underset{s\to 0}{Lt}\left[s\,G(s)H(s)\right]$$

$$= \frac{\cancel{s}\,(s+z_1)(s+z_2)(s+z_3)\ldots}{\cancel{s}\,(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= \text{Constant}$$

$\therefore$ steady state error $e_{ss} = \frac{1}{K_V} = \text{Constant}$

for type-2 Systems :-

$$K_V = \underset{s\to 0}{Lt}\left[s\,G(s)H(s)\right]$$

$$= \underset{s\to 0}{Lt}\; \cancel{s}\,\frac{(s+z_1)(s+z_2)\ldots}{s^{\cancel{2}}(s+P_1)(s+P_2)\ldots}$$

$$= \frac{1}{0} = \infty.$$

$\therefore$ steady state error $e_{ss} = \frac{1}{\infty} = \underline{\underline{0}}$.

* steady state error when the input is unit parabolic :-

generally, $e_{ss} = \lim_{s \to 0} \frac{s\,R(s)}{1+G(s)H(s)}$.

when the input is unit parabola, $R(s) = 1/s^3$

$$\therefore e_{ss} = \lim_{s \to 0} \frac{\cancel{s}\, 1/s^{\cancel{3}2}}{1+G(s)H(s)}$$

$$= \lim_{s \to 0} \frac{1}{s^2 + s^2 G(s)H(s)}.$$

$$= \frac{1}{0 + \lim_{s \to 0} s^2 G(s)H(s)}.$$

$$\therefore \boxed{e_{ss} = \frac{1}{K_a}.}$$

where, $K_a = \lim_{s \to 0} s^2 G(s)H(s)$ is known as Acceleration error Constant.

for type-0 system :-

$$K_a = \lim_{s \to 0} s^2 G(s)H(s).$$

for type-1 $\Rightarrow$ $K_a = \lim_{s \to 0} s^2 G(s)H(s)$

$$= \lim_{s \to 0} s^2 \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{(s+p_1)(s+p_2)(s+p_3)\ldots}$$

$$= 0.$$

$$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{0} = \infty.$$

Type-1 System :-

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$= \lim_{s\to 0} s^{\cancel{2}} \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{\cancel{s}(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= \underline{0}$$

$\therefore$ Steady state error $e_{ss} = \frac{1}{K_a} = \frac{1}{0}$

$$= \infty.$$

Type - 2 System :-

$$K_a = \lim_{s\to 0} s^2 G(s)H(s)$$

$$= \lim_{s\to 0} \cancel{s^2} \frac{(s+z_1)(s+z_2)(s+z_3)\ldots}{\cancel{s^2}(s+P_1)(s+P_2)(s+P_3)\ldots}$$

$$= \lim_{s\to 0} \frac{(s+z_1)(s+z_2)(s+z_3)}{(s+P_1)(s+P_2)(s+P_3)}$$

$$= \underline{\text{Constant}}$$

$\therefore e_{ss} = \frac{1}{K_a} = \frac{1}{\text{Constant}} = \underline{\text{Constant}}$

* Steady state error for various types of inputs :-

Type Number	Steady state error when input signal is-		
	Unit step	Unit Ramp	Unit Parabola.
0	$\frac{1}{1+K_p}$	∞	∞.
1	0	$\frac{1}{K_v}$	∞
2	0	0	$\frac{1}{K_a}$
3	0	0	0

* problems :-

1. obtain the response of unity feedback system whose open loop transfer function is $G(s) = \frac{4}{s(s+5)}$ and when the input is unit step signal.

Sol given transfer function $G(s) = \frac{4}{s(s+5)}$

Now, Closed loop transfer, $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$ $[\because H(s)=1]$

$$\Rightarrow C(s) = \frac{R(s)\,G(s)}{1+G(s)H(s)}$$

for unit step signal, $R(s) = \frac{1}{s}$

$$\Rightarrow C(s) = \frac{\frac{4}{s(s+5)} \cdot R(s)}{1+\frac{4}{s(s+5)}}$$

$$= \frac{4}{s(s+5)+4} R(s) = \frac{4}{s^2+5s+4} \cdot R(s)$$

$$= \frac{4}{(s+1)(s+4)} R(s)$$

$$= \frac{4}{s(s+1)(s+4)}$$

Now, by applying partial fractions,

$$\frac{4}{s(s+1)(s+4)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+4}$$

$$\Rightarrow 4 = A(s+1)(s+4) + B\,s(s+4) + C\,s(s+1),$$

Now, Substitute $s=0$

$\Rightarrow 4 = A(0+1)(0+4)$

$\Rightarrow 4 = 4A$

$\Rightarrow A = 1.$

Now, Substitute $s=-1.$

$\Rightarrow 4 = A(-1+1)(s+4) + B[-1(-1+4)]$

$\Rightarrow 4 = B(-3)$

$\Rightarrow B = \frac{-4}{3}.$

Substitute $s=-4$

$\Rightarrow 4 = C[-4(-4+1)]$

$\Rightarrow 4 = C(-4 \times -3)$

$\Rightarrow C = \frac{4}{+12} = \frac{1}{3}.$

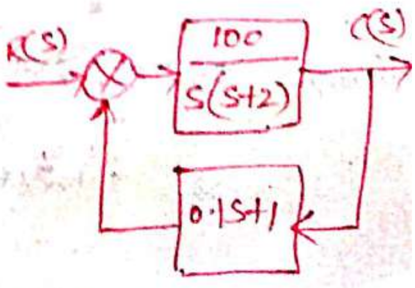
$\therefore C(s) = \frac{1}{s} - \frac{4}{3} \times \frac{1}{s+1} + \frac{1}{3} \times \frac{1}{s+4}$

Applying inverse laplace, transform we get,

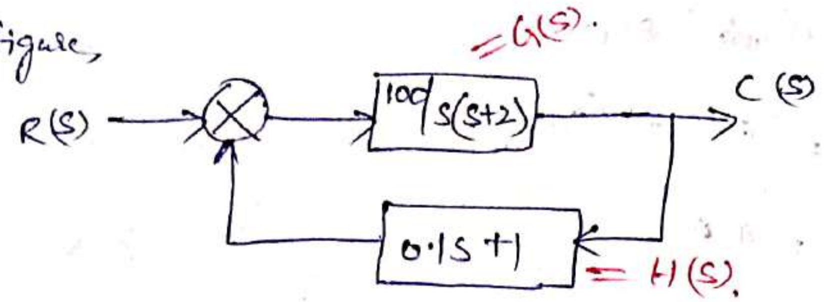
$\Rightarrow L^{-1}[C(s)] = L^{-1}\left[\frac{1}{s} - \frac{4}{3(s+1)} + \frac{1}{3(s+4)}\right]$

$\Rightarrow c(t) = 1 - \frac{4}{3}e^{-t} + \frac{1}{3}e^{-4t}$

2. A positional control system with velocity feedback is shown in fig. what is the Response of the system for unit step i/p signal.



Sol given figure,



∴ The transfer function $\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$

$$\therefore \frac{C(S)}{R(S)} = \frac{\frac{100}{S(S+2)}}{1+\frac{100}{S(S+2)}\times(0.1S+1)}$$

$$= \frac{\frac{100}{S(\cancel{S+2})}}{\frac{S(S+2)+100\times(0.1S+1)}{S(\cancel{S+2})}}$$

$$= \frac{100}{S(S+2)+10S+100}$$

$$= \frac{100}{S^2+2S+10S+100}$$

$$\therefore \frac{C(S)}{R(S)} = \frac{100}{S^2+12S+100}.$$

$$C.E = S^2+12S+100 = 0$$

$$\therefore S_1, S_2 = \frac{-12 \pm \sqrt{144-400}}{2} = \frac{-12 \pm j16}{2}$$

$$= -6 \pm j8$$

Now, equating quadratic equation to standard second order form $S^2+2\omega_n\zeta S+\omega_n^2$.

$$\Rightarrow S^2+2\omega_n\zeta S+\omega_n^2 = S^2+12S+100.$$

$\Rightarrow \omega_n^2 = 100$, $2\omega_n \xi = 12$

$\Rightarrow \omega_n = 10$, $\omega_n \xi = 6$

$$\xi = \frac{6}{10} = 0.6.$$

which lies between 0 to 1. $\Rightarrow 0 < \xi < 1$.

$\therefore$ The system is underdamped system. That means for the underdamped system the Roots are Complex & Conjugate.

Now, $$\frac{C(s)}{R(s)} = \frac{100}{s^2+12s+100}.$$

$\Rightarrow$ for unit step signal, $R(s) = \frac{1}{s}$.

$$\Rightarrow C(s) = \frac{100}{s^2+12s+100} \times R(s)$$

$$= \frac{100}{s(s^2+12s+100)}.$$

By Partial fractions,

$$\frac{100}{s(s^2+12s+100)} = \frac{A}{s} + \frac{Bs+C}{s^2+12s+100}.$$

$$\Rightarrow 100 = A(s^2+12s+100) + (Bs+C)s$$

for $s=0 \Rightarrow A = 1$.

for B and C values equating s^2 & s coefficients on both sides.

equating s^2 terms on both sides,

$$0 = A + B \Rightarrow B = -A$$

$$\therefore B = -1.$$

Equating 's' terms on both sides,

$$0 = 12A + C$$

$$\Rightarrow C = -12A$$

$$= -12$$

$$\therefore C(S) = \frac{A}{S} + \frac{(BS+C)}{s^2+12s+100}$$

$$= \frac{1}{S} + \frac{(-S-12)}{s^2+12s+100}$$

$$= \frac{1}{S} - \frac{(S+12)}{s^2+12s+100}$$

$$= \frac{1}{S} - \frac{(S+12)}{s^2+12s+36+64}$$

$$= \frac{1}{S} - \frac{(S+6)}{(S+6)^2+64} - \frac{6}{(S+6)^2+64}$$

$$= \frac{1}{S} - \frac{S+6}{(S+6)^2+8^2} - \frac{6}{8} \times \frac{8}{(S+6)^2+8^2}$$

Now, Applying inverse laplace transform,

$$\Rightarrow L^{-1}[C(S)] = L^{-1}\left[\frac{1}{S} - \frac{S+6}{(S+6)^2+8^2} - \frac{6}{8} \times \frac{8}{(S+6)^2+8^2}\right]$$

$$= 1 - e^{-6t}\cos 8t - \frac{6}{8} e^{-6t}\sin 8t.$$

$$\therefore \boxed{C(t) = 1 - e^{-6t}\left[\cos 8t + \frac{6}{8}\sin 8t\right].}$$

(3) A unity feedback control system is characterized by the following open loop transfer function $G(S) = \frac{0.4S+1}{S(S+0.6)}$. Determine the transient response for unit step signal. And also Determine

Maximum overshoot and peak time.

Let given open loop T·F $= G(s) = \dfrac{0.4s+1}{s(s+0.6)}$

for closed loop of unity feedback system.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\dfrac{0.4s+1}{s(s+0.6)}}{1+\dfrac{0.4s+1}{s(s+0.6)}}$$

$$= \frac{\dfrac{0.4s+1}{\cancel{s(s+0.6)}}}{\dfrac{s(s+0.6)+0.4s+1}{\cancel{s(s+0.6)}}} = \frac{0.4s+1}{s^2+0.6s+0.4s+1}$$

$$\therefore \frac{C(s)}{R(s)} = \frac{0.4s+1}{s^2+s+1} \Rightarrow C(s) = \frac{0.4s+1}{s(s^2+s+1)}.$$

$\left[\because \text{Unit step signal } R(s) = \frac{1}{s}.\right]$

By partial fractions,

$$\frac{0.4s+1}{s(s^2+1+s)} = \frac{A}{s} + \frac{Bs+C}{s^2+s+1}$$

$$\Rightarrow 0.4s+1 = A(s^2+s+1) + (Bs+C)s$$

for $s=0 \Rightarrow A = 1$

equating "s^2" terms we get,

$$0 = A+B$$

$$\Rightarrow B = -1.$$

Equating "s" terms on both sides,

$$0.4 = A+C$$

$$\Rightarrow -1+0.4 = C \Rightarrow C = -0.6$$

$$\therefore\ C(s) = \frac{1}{s} - \frac{(s+0.6)}{s^2+s+1}$$

$$= \frac{1}{s} - \frac{(s+0.5+0.1)}{s^2+s+0.25+0.75}$$

$$= \frac{1}{s} - \frac{s+0.5}{(s+0.5)^2+0.75} - \frac{0.1}{\sqrt{0.75}} \times \frac{\sqrt{0.75}}{(s+0.5)^2+0.75}$$

Now, applying inverse laplace transform we get,

$$c(t) = L^{-1}\left[\frac{1}{s} - \frac{(s+0.5)}{(s+0.5)^2+0.75} + \left(\frac{-0.1}{\sqrt{0.75}}\right) \times \frac{\sqrt{0.75}}{(s+0.5)^2+0.75}\right]$$

$$= 1 - e^{-0.5t}\cos\sqrt{0.75}\,t + -\frac{0.1}{\sqrt{0.75}}\,e^{-0.5t}\sin\sqrt{0.75}\,t$$

$$= 1 - e^{-0.5t}\left[\cos\sqrt{0.75}\,t + \frac{0.1}{\sqrt{0.75}}\sin\sqrt{0.75}\,t\right].$$

$$\therefore\ c(t) = 1 - e^{-0.5t}\left[\cos\sqrt{0.75}\,t + 0.115\sin\sqrt{0.75}\,t\right]$$

Now, to determine the peak overshoot & peak time,

Calculate ω_n & ξ values.

equating quadratic term to standard 2nd order system

$$\Rightarrow s^2+s+1 = s^2+2\omega_n\xi s+\omega_n^2$$

$$\Rightarrow \omega_n^2 = 1 \qquad 2\omega_n\xi = 1$$

$$\omega_n = 1 \qquad \xi = \frac{1}{2} = 0.5$$

Now, peak time, $t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

$$= \frac{\pi}{1\sqrt{1-0.5^2}} = \underline{3.628 \text{ secs}}$$

Maximum over shoot $(M_p) = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$\Rightarrow M_p = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}$$

$$= 0.163.$$

for % $M_p = 0.163 \times 100\%$.

$\therefore \% M_p = 16.3\%$..

(4) The open loop transfer function of a unity feedback type-1 system is $G(S) = \frac{K}{S(1+S)}$. For a particular value of K_1, the peak over shoot is 50%. By how much value of K_2 be increased so as to reduce the peak overshoot by half.

Sol. given open loop transfer function $G(S) = \frac{K}{S(1+S)}$.

Now, closed loop transfer function when $H(S) = 1$.

$$\Rightarrow \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

$$= \frac{\frac{K}{S(1+S)}}{1+\frac{K}{S(1+S)}} = \frac{\frac{K}{\cancel{S(1+S)}}}{\frac{S+S^2+K}{\cancel{S(1+S)}}}$$

$$= \frac{K}{S^2+S+K}$$

from the characteristic polynomial S^2+S+K.

equating CE to standard 2nd order C.E

$$\Rightarrow S^2+S+K_1 = S^2+2\omega_n\xi S+\omega_n^2$$

$$\Rightarrow \omega_n^2 = K_1 \qquad\qquad 2\omega_{n_1}\xi_1 = 1$$

$$\Rightarrow \omega_{n_1} = \sqrt{K_1} \qquad\qquad \xi_1 = \frac{1}{2\sqrt{K_1}} \quad \text{——} (1)$$

Now peak overshoot $M_p = e^{\frac{-\pi \xi_1}{\sqrt{1-\xi_1^2}}} = 0.5$

$$\frac{-\pi \xi_1}{\sqrt{1-\xi_1^2}} = \ln(0.5)$$

$$\Rightarrow \frac{\cancel{-}\pi \xi_1}{\sqrt{1-\xi_1^2}} = \cancel{-}0.7$$

$\Rightarrow$ squaring on both sides,

$$\Rightarrow \left[\frac{\pi \xi_1}{\sqrt{1-\xi_1^2}}\right]^2 = (0.7)^2$$

$$\Rightarrow (\pi \xi_1)^2 = 0.49(1-\xi_1^2)$$

$$\Rightarrow 9.86\, \xi_1^2 = 0.49 - 0.49\, \xi_1^2$$

$$\Rightarrow 9.86\, \xi_1^2 + 0.49\, \xi_1^2 = 0.49$$

$$\Rightarrow \xi_1^2 = \frac{0.49}{10.359}$$

$$\therefore \xi_1^2 = \underline{0.047}$$

To Reduce peak overshoot to half (say 0.25)

$$\Rightarrow e^{\frac{-\pi \xi_2}{\sqrt{1-\xi_2^2}}} = 0.25$$

$$\Rightarrow \frac{\cancel{-}\pi \xi_2}{\sqrt{1-\xi_2^2}} = \cancel{-}1.38$$

Squaring on both sides.

$$\frac{(\pi \xi_2)^2}{1-\xi_2^2} = (1.38)^2$$

$$\Rightarrow 9.86\, \xi_2^2 = 1.90(1-\xi_2^2)$$

$$\Rightarrow 9.86\,\xi_2^2 + 1.90\,\xi_2^2 = 1.90$$

$$\Rightarrow \xi_2^2 = \frac{1.90}{11.76}$$

$$\therefore \xi_2^2 = 0.16.$$

from eq. ① $\xi_1 = \dfrac{1}{2\sqrt{K_1}}$ similarly. $\xi_2 = \dfrac{1}{2\sqrt{K_2}}$

$$\therefore \frac{\xi_1}{\xi_2} = \frac{\cancel{2}\sqrt{K_2}}{\cancel{2}\sqrt{K_1}}$$

squaring on both sides.

$$\Rightarrow \frac{K_2}{K_1} = \frac{\xi_1^2}{\xi_2^2}$$

$$K_2 = \frac{0.047}{0.16} \times K_1$$

$$\therefore \boxed{K_2 = 0.29 \text{ times of } K_1}$$

⑤ For the open loop Transfer function given below, explain what type of input signal give rise to a constant steady state error and calculate their values.

Sol. i, $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$ ii, $G(s) = \dfrac{10}{(s+2)(s+3)}$

(iii) $\dfrac{10}{s^2(s+1)(s+2)}$.

Sol. given, i, $G(s) = \dfrac{20(s+2)}{s(s+1)(s+3)}$. [∵ Assuming unit feed back i.e, $H(s) = 1$]

Since it is a type-1 system, therefore the velocity error constant is to be calculated.

[∵ since it is having one pole at origin, so it is type-1 system]

Now, steady state error $e_{ss} = \frac{1}{K_v}$

$$\Rightarrow K_v = \lim_{s\to 0} s.\,G(s)\,H(s)$$

$$= \lim_{s\to 0} \cancel{s}\,\frac{20\,(s+2)}{\cancel{s}\,(s+1)\,(s+3)}$$

$$= \frac{20\,(0+2)}{(0+1)\;(0+3)}$$

$$= \frac{40}{3} = 13.33$$

and $e_{ss} = \frac{1}{K_v} = \frac{1}{13.33} = \underline{\underline{0.075}}$

(ii) $G(s) = \frac{10}{(s+2)\,(s+3)}$.

Since it has zero poles at origin, so, it is type-0 system. And calculate positional error constant

$$K_p = \lim_{s\to 0} G(s)\,H(s)$$

$$= \lim_{s\to 0} \frac{10}{(s+2)\,(s+3)}$$

$$= \frac{10}{2\times 3} = \frac{10}{6} = \underline{\underline{1.667}}$$

$$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+1.667}$$

$$= \underline{\underline{0.374}}$$

iii) $G(s) = \frac{10}{s^2(s+1)(s+2)}$.

It has two poles at the origin, therefore it is type-2 system. And calculating Acceleration error constant

$$K_a = \lim_{s\to 0} s^2 G(s)H(s).$$

$$= \lim_{s\to 0} \cancel{s^2} \frac{10}{\cancel{s^2}(s+1)(s+2)}$$

$$= \frac{10}{1\times 2} = 5$$

$$e_{ss} = \frac{1}{K_a} = \frac{1}{5} = 0.2$$

(6) For a unity feedback control system of open loop transfer function $G(s) = \frac{10(s+2)}{s^2(s+1)}$, find the steady state error when the input is $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$.

Sol: given, $G(s) = \frac{10(s+2)}{s^2(s+1)}$

and input $R(s) = \frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}$

the steady state error $e_{ss} = \lim_{s\to 0} s.E(s)$.

$$\Rightarrow e_{ss} = \lim_{s\to 0} s.\frac{R(s)}{1+G(s)}.$$

Unit feed back system $H(s) = 1$

$$E_s = \frac{R(s)}{1+G(s)}.$$

$$= \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{1 + \frac{10(s+2)}{s^2(s+1)}} = \frac{\frac{3}{s} - \frac{2}{s^2} + \frac{1}{3s^3}}{\frac{s^2(s+1) + 10s + 20}{s^2(s+1)}}.$$

$$\Rightarrow E(S) = \frac{3}{\cancel{s}} \frac{s^{\cancel{2}}(S+1)}{10(S+2)+s^2(S+1)} - \frac{2}{\cancel{s^2}} \frac{\cancel{s^2}(S+1)}{10(S+2)+s^2(S+1)} + \frac{1}{3s^3} \frac{\cancel{s^2}(S+1)}{s^2(S+1)+10(S+2)}.$$

$$\Rightarrow E(S) = 3\frac{S(S+1)}{s^2(S+1)+10S+20} - \frac{2(S+1)}{s^2(S+1)+10S+20} + \frac{1}{3S}\left(\frac{(S+1)}{s^2(S+1)+10S+20}\right)$$

$\therefore$ Steady state error $e_{ss} = \lim_{S\to 0} S \cdot E(S)$.

$$\Rightarrow \lim_{S\to 0} S \cdot \left[\frac{3\,S(S+1)}{s^2(S+1)+10S+20} - \frac{2(S+1)}{s^2(S+1)+10S+20} + \frac{1}{3S}\frac{(S+1)}{s^2(S+1)+10S+20}\right]$$

$$\Rightarrow \lim_{S\to 0} \frac{s^2 3(S+1)}{s^2(S+1)+10S+20} - \lim_{S\to 0} \frac{2S(S+1)}{s^2(S+1)+10S+20} + \lim_{S\to 0} \frac{\cancel{s}(S+1)}{3\cancel{s}[s^2(S+1)+10S+20]}$$

$$\Rightarrow e_{ss} = 0 - 0 + \frac{(0+1)}{0(0+1)+0+(20)3}$$

$$\Rightarrow e_{ss} = \frac{1}{20\times 3} = \frac{1}{60}$$

$\therefore$ The steady error of $G(S) = \frac{10(S+2)}{s^2(S+1)}$ for $R(S) = \frac{3}{S} - \frac{2}{s^2} + \frac{1}{3s^3}$ is $\frac{1}{20\times 3} = \frac{1}{60}$.

(7) A unit feedback system is characterised by an open-loop transfer function $G(S) = \frac{K}{S\cdot(S+10)}$. Determine the K value so that system will have damping ratio 0.5. And for this value of 'K'. Determine rise time, peak time and over shoot.

Sol: given $G(s) = \frac{K}{s(s+10)}$,

for unity closed loop transfer function $H(s) = 1$, $\xi = 0.5$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$= \frac{\frac{K}{s(s+10)}}{1+\frac{K}{s(s+10)}}$$

$$= \frac{K}{s^2+10s+K}$$

Now, Equating $s^2+10s+K=0$ to ~~s~~ standard second order

system $s^2+2\omega_n\xi s+\omega_n^2=0$

$$\Rightarrow s^2+10s+K = \cancel{10s+K+s^2}\ s^2+\omega_n^2+2\xi\omega_n s$$

$\Rightarrow \omega_n^2 = K$

$\Rightarrow \omega_n = \sqrt{K}$

$2\omega_n\xi = 10$

$\omega_n\xi = 5$

$\omega_n = 5/\xi = \frac{5}{0.5} = 10.$

$$\Rightarrow \sqrt{K} = 10.$$

$$\Rightarrow K = 10^2 = 100.$$

$$\therefore \frac{C(s)}{R(s)} = \frac{K}{s^2+10s+100}$$

$$\Rightarrow \frac{C(s)}{R(s)} = \frac{100}{s^2+10s+100}.$$

Now, $\omega_n^2 = 100$

$\Rightarrow \omega_n = 10$

$2\xi\omega_n = 10$

$\Rightarrow \xi\omega_n = 5$

$\Rightarrow \xi = 5/10 = 0.5$

Now, ~~Rise~~ Peak time $t_p = \frac{\pi}{\omega_d}$

$$\Rightarrow t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi}{10\sqrt{1-0.5^2}}$$

$$\boxed{\therefore t_p = 0.362 \text{ secs.}}$$

Rise ~~Peak~~ time ~~t_p~~ $t_r = \frac{\pi-\theta}{\omega_d}$

$$= \frac{\pi-\theta}{\omega_n\sqrt{1-\xi^2}}$$

$$t_r = \frac{\pi-\theta}{10\sqrt{1-\xi^2}}$$

$$\left[\because \theta = \tan^{-1}\left(\frac{\sqrt{1-\xi^2}}{\xi}\right) = \tan^{-1}\left(\frac{0.866}{0.5}\right) = 60^\circ\right]$$

Now, $\theta = 60^\circ$ Radians

Converting Radians to degrees :-

$$\boxed{1 \text{ Radian} = \frac{180}{\pi} \text{ deg}}$$

Similarly, degrees to Radians

$$\boxed{1 \text{ deg} = \frac{\pi}{180} \text{ Radian}}$$

$$\therefore 60^\circ = 60\times\frac{\pi}{180}$$

$$= 1.047 \text{ rad}$$

$$\therefore t_r = \frac{\pi - 1.047}{10\sqrt{1-0.5^2}} = \frac{\pi - 1.047}{8.66}$$

$$\therefore \boxed{t_r = 0.241 \text{ secs.}}$$

Maximum peak over shoot $M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$

$$\Rightarrow M_p = e^{\frac{-0.5\pi}{\sqrt{1-\xi^2}}}$$

$$= e^{-0.5\pi/\sqrt{1-0.5^2}}$$

$$= e^{\frac{-1.57}{0.866}}$$

$$= e^{-1.8129}$$

$$= 0.163 \text{ Sec.}$$

(8) For unity feedback system having open loop transfer function as $G(s) = \frac{K(s+2)}{s^2(s^2+7s+12)}$. Determine (i) type of a system (ii) Error constants K_p, K_v, K_a. (iii) steady state error for unit parabolic input.

Sol. given, $G(s) = \frac{K(s+2)}{s^2(s^2+7s+12)}$.

type of the system $=2$

Since, the 2 Roots are lying at the origin therefore, decides the type of a system.

(ii) Error constants $K_p = \lim_{s \to 0} G(s)$.

$$= \lim_{s \to 0} \frac{K(s+2)}{s^2(s^2+7s+12)}$$

$$= \frac{K(0+2)}{0}$$

$$= \infty$$

$$K_v = \lim_{s\to 0} s \cdot G(s)$$

$$= \lim_{s\to 0} \cancel{s} \cdot \frac{K(s+2)}{s^{\cancel{2}}(s^2+7s+12)}$$

$$= \frac{K(0+2)}{0(0+0+12)}$$

$$= \infty.$$

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$$= \lim_{s\to 0} \cancel{s^2} \frac{K(s+2)}{\cancel{s^2}(s^2+7s+12)}$$

$$= \frac{K(0+2)}{0+0+12}$$

$$= \frac{2K}{12} = \frac{K}{6}.$$

iii, Error Constant when input is parabolic is $K_a = \frac{K}{6}$

Now, Steady state error $e_{ss} = \frac{1}{K_a}$

$$\Rightarrow e_{ss} = \frac{1}{\frac{K}{6}}$$

$$\therefore \boxed{e_{ss} = \frac{6}{K}}$$

(9) Find the position, Velocity and Acceleration Constants for the following unity feedback system having forward loop transfer function as,

i, $G(s) = \frac{K}{s^2(s^2+8s+100)}$ ii, $\frac{K(1+s)(1+2s)}{s^2(s^2+4s+20)}$

Sol. i, given $G(s) = \frac{K}{(s^2+8s+100)s^2}$

Now, $K_a = \lim_{s\to 0} s^2 G(s)$

$= \lim_{s\to 0} \cancel{s^2} \frac{K}{\cancel{s^2}(s^2+8s+100)}$

$= \frac{K}{100}$

$$\boxed{e_{ss} = \frac{1}{K_a} = \frac{100}{K}}$$

$K_v = \lim_{s\to 0} s \cdot G(s)$

$= \lim_{s\to 0} \cancel{s} \cdot \frac{K}{s^2(s^2+8s+100)}$

$= \lim_{s\to 0} \frac{K}{s(s^2+8s+100)}$

$= \frac{K}{0} = \infty$

$\therefore e_{ss} = \frac{1}{\infty} = 0.$

$K_p = \lim_{s\to 0} G(s)$

$= \lim_{s\to 0} \frac{K}{s^2(s^2+8s+100)}$

$= \frac{K}{0} = \infty$

$e_{ss} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0.$

ii) $G(s) = \frac{K(1+s)(1+2s)}{s^2(s^2+4s+20)}$

$K_p = \lim_{s\to 0} G(s) = \lim_{s\to 0} \frac{K(1+s)(1+2s)}{s^2(s^2+\cancel{8}4s+20)}$

$= \frac{K(1+s)(1+2s)}{0} = \infty$

$e_{ss} = \frac{1}{1+K_p} = \frac{1}{1+\infty} = 0$

$$K_v = \lim_{s\to 0} s \cdot G(s)$$

$$= \lim_{s\to 0} \cancel{s} \frac{K(1+s)(1+2s)}{s^{\cancel{2}}(s^2+8s+20)}$$

$$\therefore K_v = \frac{K(1\times 1)}{0} = \infty$$

$$K_a = \lim_{s\to 0} s^2 \cdot G(s)$$

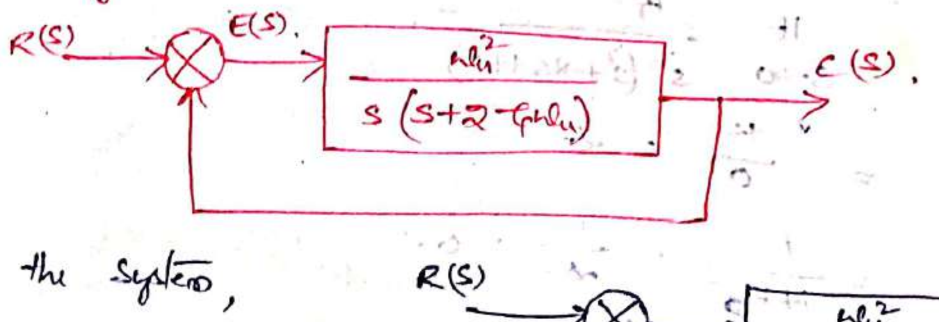
$$= \lim_{s\to 0} \cancel{s^2} \frac{K(1+s)(1+2s)}{\cancel{s^2}(s^2+8s+20)}$$

$$= \frac{K(1+0)(1+0)}{0+0+20}$$

$$\therefore \boxed{K_a = \frac{K}{20}}$$

$$\therefore \boxed{e_{ss} = \frac{1}{K_a} = \frac{20}{K}}$$

(10) Consider a system for $\zeta = 0.5$ and $\omega_n = 0.5$ shown in fig. Determine Rise time, peak time, maximum overshoot and settling time, when the system is unit step.



Sol. for the system,

$$\Rightarrow \frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)}$$

$$= \frac{\dfrac{\omega_n^2}{s^2+2\omega_n \zeta s}}{1+\dfrac{\omega_n^2}{s^2+2\omega_n \zeta s}}$$

$$\therefore \frac{C(S)}{R(S)} = \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

where, $\zeta = 0.6^5$, $\omega_n = 0.5$

Now, Rise time, $t_r = \frac{\pi - \theta}{\omega_d}$

$\omega_d = (\sqrt{1-\zeta^2})\omega_n = \sqrt{1-0.5^2} \times 0.5$
$= 0.866 \times 0.5 = 0.433$

$\theta = \tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$

$= \tan^{-1}\left(\frac{0.866}{0.5}\right) = 60^\circ.$

$\Rightarrow$ Converting degrees to Radians.

$\Rightarrow 60^\circ = 60 \times \frac{\pi}{180}$

$= 60 \times 0.0174 = 1.047$ rad.

$\therefore t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.047}{(0.866 \times 0.5)}$

$= 4.837$ secs.

peak time, $t_p = \frac{\pi}{\omega_d}$

$= \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$

$= \frac{\pi}{0.5\sqrt{1-0.5^2}}$

$= 7.255$ secs.

Peak overshoot $M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$

$\Rightarrow M_p = e^{\frac{-0.5\pi}{\sqrt{1-0.5^2}}}$

$= e^{-1.813} = \underline{\underline{0.163}}.$

Settling time for 2% error

$$t_s = \frac{4}{\xi \omega_n}$$

$$= \frac{4}{0.5 \times 0.5}$$

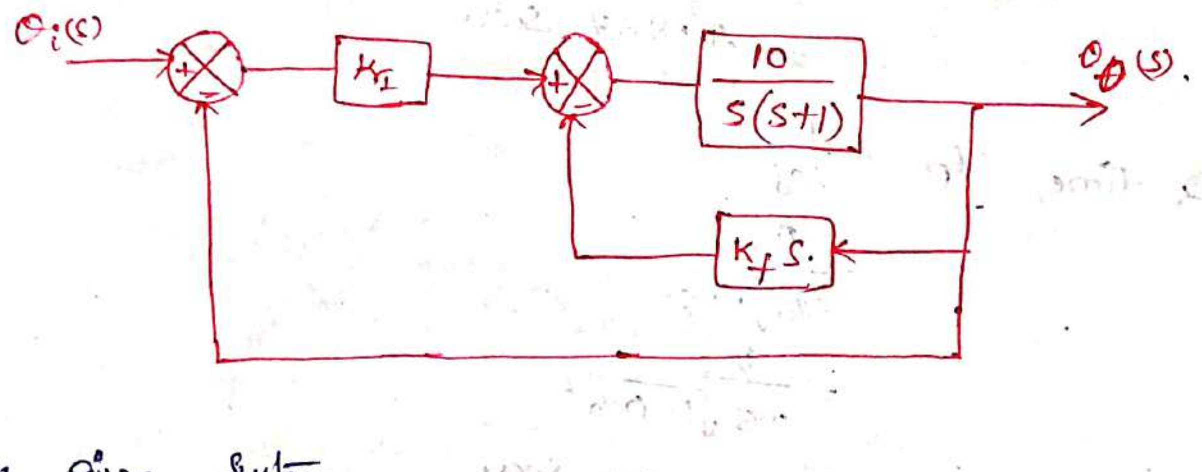
$$= 16 \text{ sec.}$$

for 5% error,

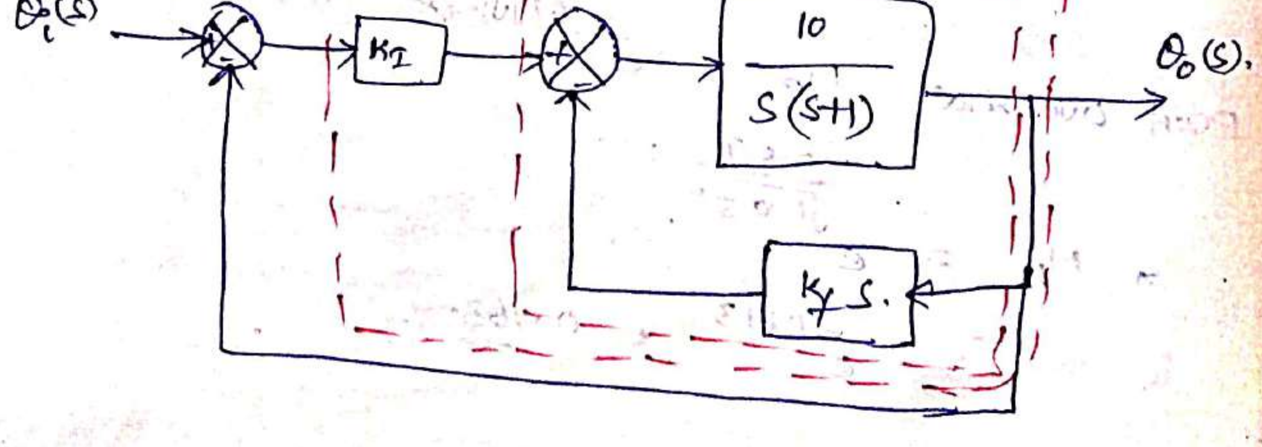
$$t_s = \frac{3}{\xi \omega_n}$$

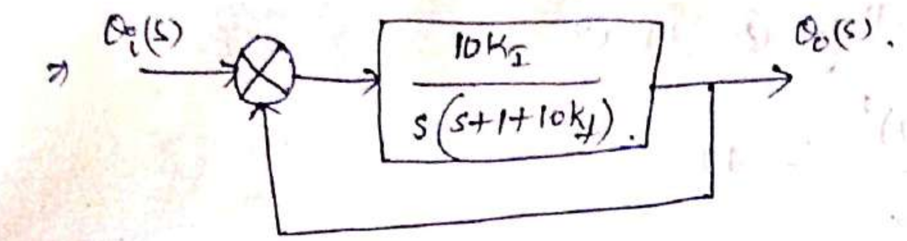
$$= \frac{3}{0.5 \times 0.5} = 12 \text{ sec.}$$

(11) The system as shown in fig. has following specifications $K_v = 10$, $\xi = 0.5$. Find the value of K_I and K_f to meet the given specifications of the system.



Sol. given system





$$\therefore T.F = \frac{\theta_o(s)}{\theta_i(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{10K_I}{s^2+(10K_f+1)s+10K_I}$$

The open loop transfer function of a system

$$G(s) = \frac{10K_I}{s(s+1+10K_f)}$$

Now, given, $K_v = 10$

$$\Rightarrow K_v = \lim_{s\to 0} s.G(s)$$

$$= \lim_{s\to 0} \cancel{s}. \frac{10K_I}{\cancel{s}(s+1+10K_f)}$$

$$= \frac{10K_I}{10K_f+1} = 10. \quad \text{——} (1).$$

Now, by Comparing the C.E $s^2+(10K_f+1)s+10K_I$ to 2nd order standard equation,

$$\Rightarrow s^2+s(10K_f+1)+10K_I = s^2+2\omega_n\xi s+\omega_n^2$$

$$\Rightarrow \omega_n^2 = 10K_I.$$

$$2\xi\omega_n = 10K_f+1$$

$$2\times 0.5\,\omega_n = 10K_f+1$$

$$\Rightarrow \omega_n = 10K_f+1.$$

$$\Rightarrow \sqrt{10K_I} = 10K_f+1$$

$$\Rightarrow 10K_I = (10K_f+1)^2. \quad \text{——} (2)$$

Now, Substituting ② in ①

$$\Rightarrow \frac{(10k_f+1)^2}{\cancel{10k_f+1}} = 10$$

$$\Rightarrow 10k_f = 9$$

$$\Rightarrow k_f = \frac{9}{10} = 0.9.$$

from eq. ①.

$$\Rightarrow \frac{10k_I}{10k_f+1} = 10$$

$$\Rightarrow \frac{10k_I}{10\times0.9+1} = 10$$

$$\Rightarrow \frac{\cancel{10}k_I}{\cancel{9+1}} = 10$$

$$\Rightarrow \therefore k_I = 10.$$

(12) A unity feedback system with $G(s) = \frac{K(s+\alpha)}{(s+\beta)^2}$ is to be designed to meet the following specifications. Steady state error for a unit step input $= 0.1$, Damping Ratio $= 0.5$ and Natural frequency $\omega_n = \sqrt{10}$.

Sol. given $G(s) = \frac{K(s+\alpha)}{(s+\beta)^2}$ $[\because H(s) = 1]$.

e_{ss} for unit step $= 0.1$

$\xi = 0.5$, $\omega_n = \sqrt{10}$.

The steady state error $e_{ss} = \underset{s\to0}{Lt}\, s.\frac{R(s)}{1+G(s)H(s)}$.

for unit step signal $R(S) = 1/s$.

$$\Rightarrow e_{ss} = \lim_{s\to 0} \frac{\cancel{s}\,\frac{1}{\cancel{s}}}{1+\frac{K(s+\alpha)}{(s+\beta)^2}} = 0.1$$

$$= \lim_{s\to 0} \frac{1}{1+\frac{K(s+\alpha)}{(s+\beta)^2}} = 0.1$$

Now apply limits

$$\Rightarrow e_{ss} = \frac{1}{1+\frac{K\alpha}{\beta^2}} = 0.1$$

$$= 1+\frac{K\alpha}{\beta^2} = 10$$

$$\Rightarrow \frac{K\alpha}{\beta^2} = 10-1 = 9$$

$$\Rightarrow \boxed{K\alpha = 9\beta^2} \quad \text{--- (1)}$$

The closed loop Transfer function is given by

$$\frac{C(S)}{R(S)} = \frac{G(S)}{1+G(S)H(S)}$$

$$= \frac{\frac{K(s+\alpha)}{(s+\beta)^2}}{1+\frac{K(s+\alpha)}{(s+\beta)^2}} = \frac{K(s+\alpha)}{s^2+\beta^2+2\beta s+Ks+K\alpha}$$

$$\therefore \frac{C(S)}{R(S)} = \frac{K(s+\alpha)}{s^2+(2\beta+K)s+K\alpha+\beta^2}$$

Now Equating the quadratic term to standard form of second order equation.

$\Rightarrow s^2 + (2\beta + K)s + K\alpha + \beta^2 = s^2 + 2\omega_n \zeta s + \omega_n^2$

$\Rightarrow \omega_n^2 = K\alpha + \beta^2$

$\Rightarrow (\sqrt{10})^2 = K\alpha + \beta^2$

$\Rightarrow K\alpha + \beta^2 = 10$ — ②

$2\omega_n \zeta = 2\beta + K$

$2 \times \sqrt{10} \times 0.5 = 2\beta + K$

$\Rightarrow 2\beta + K = \sqrt{10}.$ — ③

Now, Substituting eq. ① in ②

$\Rightarrow 9\beta^2 + \beta^2 = 10$

$\Rightarrow \beta^2 = \frac{10}{10} = 1$

$\Rightarrow \beta = \pm 1.$

when $\beta = +1$ then

③ $\Rightarrow 2\beta + K = \sqrt{10}$

$\Rightarrow K = \sqrt{10} - 2$

$\boxed{\therefore K = 1.162.}$

② $\Rightarrow K\alpha + \beta^2 = 10$

$\Rightarrow \alpha 1.162 + \beta^2 = 10$

$\Rightarrow \alpha = \frac{10 - 1}{1.162}$ $[\because \beta^2 = 1]$

$= \frac{9}{1.162}$

$\therefore \boxed{\alpha = 7.745}$

when $\beta = -1$ then.

③ $\Rightarrow 2\beta + K = \sqrt{10}$

$\Rightarrow 2 \times -1 + K = \sqrt{10}$

$\Rightarrow K = \sqrt{10} + 2$

$= 3.162 + 2$

$\therefore \boxed{K = 5.162}$

② $\Rightarrow K\alpha + \beta^2 = 10$

$\Rightarrow 5.162\alpha + \beta^2 = 10$

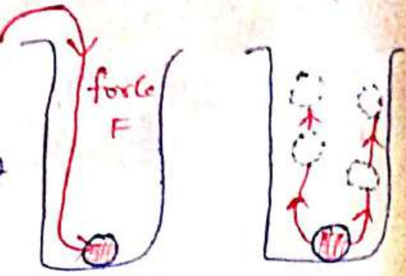
$\Rightarrow \alpha = \frac{10 - 1}{5.162}$

$\therefore \boxed{\alpha = 1.743}$

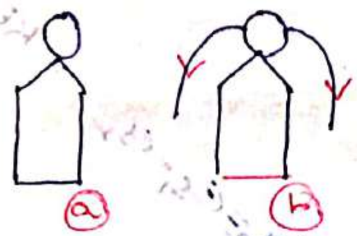
Unit IV

* Concept of Stability :-

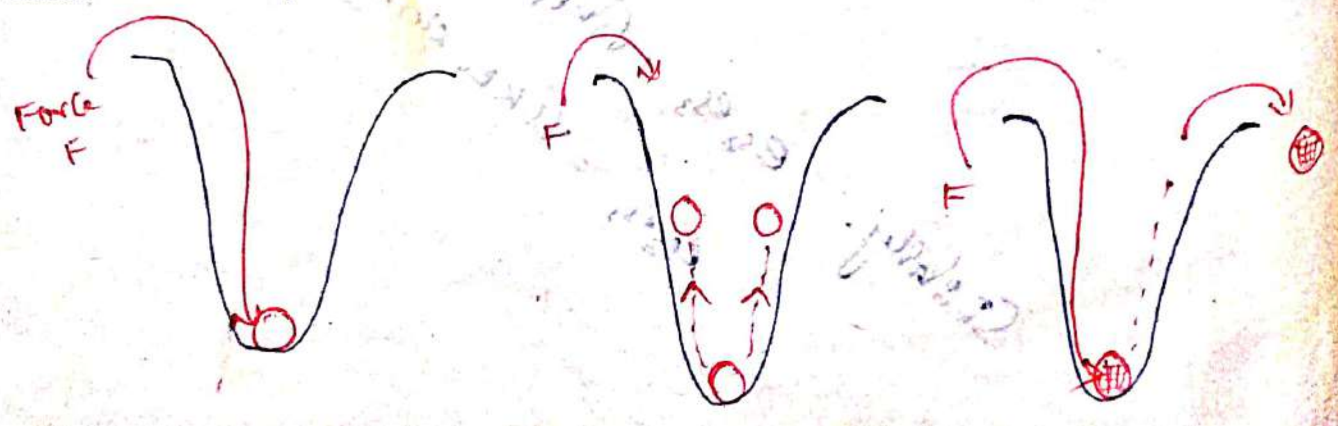
1. Consider a System i.e a deep Container with an object placed inside it as shown in figure. Now, if we apply a force to take out the object, as the depth of the Container is more, it will oscillate and well settle down again as its original position. If we assume that the force required to take out the object tends to infinity i.e always object will oscillate when force is applied and well settle down but not come out, Such a System is called Absolutely stable.



2. Consider a Container which is pointed one, on which we try to keep a circular object. In this case, object will fall without any external Disturbance of force. So, if we try to keep the circular object, we will fail always to do so. Such a System is called unstable System.



3. while in certain cases the Container is shallow then there exists a critical value of force for which object will come out of Container.



As long as $F < F_{critical}$, object Regains its original position but if $F > F_{critical}$, object will Come out. Stability depends on Certain Conditions of the system hence system is Called Conditionally stable system.

* Routh - Hurwitz Criterion :-

This Represents a Method of Determining the location of poles of a characteristic equation with Respect to the left half and Right half of the s-plane without actually solving the equation.

The T.F of any linear closed loop system can be Represented as,

$$\boxed{\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n} = \frac{B(s)}{F(s)}}$$

where a and b are Constants.

To find the closed loop poles we equate $F(s) = 0$. This equation is Called the characteristic equation of the system.

i.e $\Rightarrow F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0.$

Thus, the Roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

* Determine the stability of the given characteristic equation by Hurwitz method.

$$F(s) = s^3 + s^2 + s^1 + 4 = 0.$$

Sol. $F(S) = s^3+s^2+s^1+4=0$.

$a_0=1,\ a_1=1,\ a_2=1,\ a_3=4$.

Highest order $n=3$.

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}_{3\times3} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

D_1, D_2, $\rightarrow D_3$.

$|D_1| = |1| = 1$

$|D_2| = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = -3$.

$|D_3| = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = -12$

[∵ The number of roots will be determined by the highest degree of C.E.]

As D_2 and D_3 are negative, given system is unstable.
As there is one sign change so, one root lies in the right half of s-plane and the Remaining two lies in left half of the s-plane.

Disadvantages:-

1. For Higher order Systems, to solve the determinants of higher order is very Complicated and time consuming.
2. Difficult to predict marginal stability of the system.
3. For the system to be stable, all the above determinants must be positive.

* Routh's stability criterion :-

It is also called as Routh's array Method or Routh's Hurwitz's Method.

Routh suggested a method of tabulating the Coefficients of characteristic equation in a particular way. Tabulation of the coefficients gives an array called Routh's Array.

Consider the general characteristic equation as,

$$F(s) = a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots\dots + a_n = 0.$$

Method of Forming an array :-

$$\begin{array}{c|cccc}
s^n & a_0 & a_2 & a_4 & a_6 \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 \\
s^{n-2} & b_1 & b_2 & b_3 & \\
s^{n-3} & c_1 & c_2 & c_3 & \\
\vdots & \vdots & & & \\
s^0 & a_n. & & &
\end{array}$$

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1}.$$

$$c_3 = \frac{b_1 a_7 - a_1 \times 0}{b_1.}$$

*** Routh's Criterion :-**

The Routh's stability criterion can be stated as,

"The necessary and sufficient condition for stability is that all of the elements in the first column of the Routh Array be positive. If this condition is not met, the system is unstable and the number of sign changes in the elements of the first column of the Routh array corresponds to the number of roots of the characteristic equation in the right half of s-plane".

In the process of constructing the Routh Array the missing terms are considered as zeros. Also, all the elements of any row can be multiplied (or) Divided by a positive constant to simplify the computational work.

In the Construction of Routh Array one may Come across the following 3 Cases.

Case-I :- Normal Routh Array (Non-zero elements in the first Column of Routh Array).

Case-II :- A row of all zeros.

Case-III :- First element of a row is zero but some (or) other elements are not zero.

Case-I :- Normal Routh Array :-

In this Case there is no difficulty in forming Routh Array. The Routh Array can be constructed as explained above. The sign changes are noted to find the number of roots lying on the right half of s-plane and the stability of the System can be estimated.

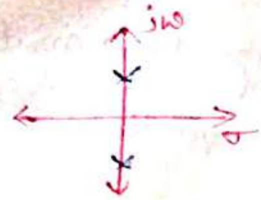
In this Case if there is no sign changes in the first Column of Routh array then all the roots are lying on the left half of s-plane and the System is stable

If there is sign change in the first Column of Routh array, then the System is unstable and the number of roots lying on the Right half of s-plane is equal to the number of sign changes. The Remaining roots are on left half of s-plane

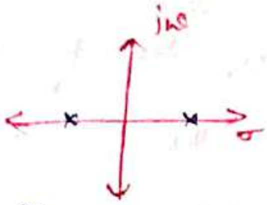
Case-II :- A Row of All Zeros :-

An all zero row indicates the existance of an even polynomial as a factor of the given characteristic equation. This even polynomial factor is Called the "Auxillary polynomial". The Co-efficients of Auxillary polynomial will always be elements of row directly above the row of zeros ~~in~~ in the array.

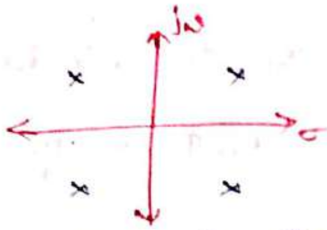
The roots of even polynomial occurs in pairs that are equal in magnitude and opposite in sign. Hence these roots can be purely imaginary, purely Real (or) Complex.



ⓐ purely imaginary.



ⓑ purely Real



ⓒ Complex Roots.

Method - 1 :-

1. Determine the Auxillary polynomial A(s).
2. Differentiate the Auxillary polynomial with respect to s to get $\frac{d\,A(s)}{ds}$.
3. A Row of zeros is Replaced with coefficients of $\frac{dA(s)}{ds}$.
4. Continue the Construction of the array in the usual manner.

ⓐ If there is a ~~sing~~ sign changes in the first coloumn of the Routh Array then the System is unstable. The number of roots lying on the right half of s-plane is equal to the number of sign changes. The number of Roots lying on the imaginary axis can be estimated from the roots of auxillary polynomial. The Remaining roots are lying on the left half of s-plane.

ⓑ If there is no sign changes in the first coloumn of the routh array then all zero row indicate the existance of purely imaginary roots and so the System is "limitedly" (or) "Marginally stable"

The roots of auxillary equation lies on the imaginary axis and the remaining roots lies on the left-half of S-plane

Method-2 :-

1. Determine the auxillary polynomial A(S).
2. Divide the characteristic equation by Auxillary polynomial.
3. Construct Routh array using the Coefficients of quotient polynomial.
4. The array is intepreted as follows.

ⓐ If there is a sign change in the first coloumn of Routh array of quotient polynomial then the system is unstable. The number of roots of quotient polynomial lying on right half of S-plane is given by number of sign changes in first coloum.

The roots of auxillary polynomial are directly calculated to find whether they are purely imaginary (or) purely Real (or) Complex.

The total number of roots on the Right half of S-plane is given by the Sum of number of sign changes and the number of roots of auxillary polynomial with positive Real part. The number of roots on imaginary axis can be estimated from the roots of auxillary polynomial. The remaining roots are lying on the left half of S-plane.

ⓑ If there is no sign change in the first coloumn of the Routh Array of quotient polynomial then the signal is limitedly (or) marginally stable, since there is no sign change all the roots of quotient polynomial are lying on the left half of splane

The roots of auxillary polynomial are directly calculated to find whether they are purely imaginary (or) Real (or) Complex. The number of roots lying on imaginary axis and on the right half of S-plane can be estimated from the roots of auxillary polynomial. The Remaining roots are lying on the left half of s-plane

Case - III . First Element of a Row is zero :-

while constructing a Routh Array, if a zero is encountered as first element of a row then all the elements of a next row will be infinite. To overcome this problem let $0 \rightarrow \varepsilon$ and complete the construction of array in the usual way.

Finally let $\varepsilon \rightarrow 0$ and determine the values of the elements of the array which are functions of ε. The Resultant array is intepreted as follows.

(a) If there is no sign change in the first coloumn of the Routh array and if there is no row with all zeros, then all the roots are lying on the left half of s-plane and the system is stable.

(b) If there is a sign changes in the first coloumn of Routh array and there is no row with all zeros, then some of the roots are lying on the right half of s-plane and the system is unstable. The number of roots lying on the right half of s-plane is equal to number of sign changes and the Remaining roots are lying on the left half of s-plane.

(c) If there is a row of all zeros after letting $\varepsilon \to 0$, then there is a possibility of roots on imaginary axis. Determine the auxillary polynomial and divide the characteristic equation by auxillary polynomial to eliminate the imaginary roots. The Routh Array is constructed using the coefficients of quotient polynomial and the characteristic equation is intepreted as explained in Method-2 of Case-II polynomial.

*** Problems:-**

1. Using Routh criterion, determine the stability of system represented by the characteristic equation, $s^4+8s^3+18s^2+16s+5=0$. Comment the location of roots of C.E

Sol: given C.E $\Rightarrow s^4+8s^3+18s^2+16s+5=0$.

The given characteristic equation is of 4th order and so, it has 4 roots. Since, the highest power is even number, first the row of routh array starts with the coefficients of even powers and from the second row using the coefficients of odd powers of s.

$\Rightarrow$

s^4	: 1	18	5
s^3	: 1	2	
s^2	: 16	5	
s^1	: 1.7		
s^0	: 5		

$$s^2 = \frac{1\times18-2\times1}{1}, \quad \frac{1\times5-0\times1}{1}$$
$$= 16. \qquad 5$$
$$\therefore s^2 : 16 \quad 5$$
$$s^1 = \frac{16\times2-5\times1}{16}$$
$$= 1.6875 \cong 1.7$$
$$s^0 = \frac{1.7\times5-0\times16}{1.7}$$
$$= 5.$$
$$\therefore s^0 : 5$$

It is observed that all the elements of the Routh Array are positive and there is no sign change. Hence all the roots are lying on the left half of s-plane and the system is stable.

Result :

1. The system is stable
2. Since, no sign change, all the roots are lying on left half of the s-plane.

(2) $s^3+6s^2+11s+6=0$.

Sol: order $=3$, therefore, 3 roots would be obtained.

$$\Rightarrow \begin{array}{lll} s^3 & : 1 & 11 \\ s^2 & : 6 & 6 \\ s^1 & = 10 & \\ s^0 & : 6. & \end{array}$$

$s^1 : \dfrac{6\times 11 - 1\times 6}{6} = \dfrac{60}{6} = 10$,

$s^0 : \dfrac{10\times 6 - 6\times 0}{10} = 6$

∴ The system is stable since there is no sign change in the first column of Routh Array table. Therefore, all the roots are lying on the half (left) of s-plane.

3. $s^5+2s^4+6s^3+3s^2+2s+1=0$.

Sol: The highest order of given characteristic equation is 5. So, it should contains 5 roots.

$$\Rightarrow \begin{array}{llll} s^5 & : 1 & 6 & 2 \\ s^4 & : 2 & 3 & 1 \\ s^3 & : 4.5 & 1.5 & \\ s^2 & : 2.33 & 1 & \\ s^1 & : -0.429 & & \\ s^0 & : 1 & & \end{array}$$

∴ There is two sign changes in the first coloumn of Routh Array table. So, the system is unstable.

→ As there are two sign changes, the two roots are lying on the right half of s-plane and the Remaining roots are lying on the left half of s-plane.

4. $8s^5+4s^4+30s^3+25s^2+62s-44=0$.

Sol. The given characteristic Equation $= 8s^5+4s^4+30s^3+25s^2+62s-44=0$

As the highest order of C.E is 5 so, it contains 5 roots.

The highest power of C.E is of odd number so, the Routh Array starts with odd power coefficients in decreasing order.

s^5 :	8	30	62
s^4 :	4	25	−44
s^3 :	−20	150	
s^2 :	55	−44	
s^1 :	134		
s^0 :	−44.		

$s^3 : \frac{4\times 30-25\times 8}{4}, \frac{4\times 62+8\times 44}{4}$

$-20, 150.$

$s^2 : \frac{-20\times 25-150\times 4}{-20}, \frac{-20\times -44}{-20}$

$= \frac{-1100}{-20}, -44.$

$= 55$

$s^1 : \frac{55\times 150-(-20\times -44)}{55}$

$= 134$

$s^0 = \frac{134\times -44}{134} = -44.$

As there are 3 sign changes in the first coloumn of the Routh Array table. So, the system is unstable.

As there are 3 sign changes in first coloumn thus, the 3 roots are lying on the right half of s-plane and Remaining 2 roots lies on the left half of s-plane.

Under Case - III :

Problems:-

1. Determine the stability of the system for the given C.E.

$s^4 + 2s^3 + 10s^2 + 20s + 5 = 0.$

Sol: The Routh table can be formed as,

$s^4 \quad : \quad 1 \quad 10 \quad 5$

$s^3 \quad : \quad 2 \quad 20.$

$s^2 \quad : \quad \frac{2\times10 - 20\times1}{2} = 0 \quad \frac{2\times5 - 1\times0}{2} = 5$

$s^2 \quad : \quad 0 \quad 5$

The first element in the s^2 is zero, whereas the non-zero element in the same row. So, the system is unstable. To find the number of roots in the right half of s-plane, replace the first zero element by a small positive integer 'ε'. and proceed with the Routh array formation.

$\therefore$

$s^4 \quad : \quad 1 \quad 10 \quad 5$

$s^3 \quad : \quad 2 \quad 20$

$s^2 \quad : \quad \varepsilon \quad 5$

$s^1 \quad : \quad \frac{\varepsilon\times20 - 2\times5}{\varepsilon} \Rightarrow -\infty$

$s^1 \quad : \quad -\infty$

$s^0 \quad : \quad 5$

$s^4 \quad : \quad 1 \quad 10 \quad 5$

$s^3 \quad : \quad 2 \quad 20$

$s^2 \quad : \quad 0 \quad 5$

$s^1 \quad : \quad -\infty$

$s^0 \quad : \quad 5$

Now, Replace the $\varepsilon \to 0$ in above array. As there are two sign changes. So, the two roots are lying on the right half of s-plane and the Remaining roots are lying on the left half of s-plane..

Alternative Method :- for C.E $s^6+2s^5+s^4+2s^3+4s+3s^2+5=0$.

Replace s in the characteristic equation by 1/z and transform the problem in the s-plane to the z-plane. The characteristic equation in the z-plane is,

$$\Rightarrow \left(\frac{1}{z}\right)^6+2\left(\frac{1}{z}\right)^5+\left(\frac{1}{z}\right)^4+2\left(\frac{1}{z}\right)^3+4\cdot\left(\frac{1}{z}\right)+3\left(\frac{1}{z}\right)^2+5=0$$

$$\Rightarrow 5z^6+4z^5+3z^4+2z^3+z^2+2z+1=0.$$

Now, construct the Routh Array table for C.E in z-plane.

s^6 :	5	3	1
s^5 :	4	2	2
s^4 :	0.5	-1.5	1
s^3 :	14	-6	
s^2 :	-1.285	1	
s^1 :	4.88	0	
s^0 :	1		

from this method ~~also~~, we found two sign changes in the first column of Routh Array table. So, two roots are lies on the right half of s-plane and the remaining roots are lies on the left half of s-plane.

2. Main Method :-

given characteristic equation, $s^6+2s^5+s^4+2s^3+3s^2+4s+5=0$

As, the order of the given characteristic equation is 6. So, it contains totally 6 Roots. The Routh table is formed as follows

$$
\begin{array}{llll}
s^6 : & 1 & 1 & 3 & 5 \\
s^5 : & 2 & 2 & 4 & \\
s^4 : & 0 & 1 & 5 & \\
s^3 : & & & &
\end{array}
$$

The fixet element of s^4 row is zero. So, the system is unstable. Now, Replacing the value zero with the positive integer. ε.

$\therefore$

$$
\begin{array}{lll}
s^6 : & 1 & 1 & 3 & 5 \\
s^5 : & 2 & 2 & 4 & \\
s^4 : & \varepsilon & 1 & 5 & \\
s^3 : & \dfrac{\varepsilon \times 2 - 2\times 1}{\varepsilon} & \dfrac{4\varepsilon-10}{\varepsilon} & & \\
s^2 : & \dfrac{-4\varepsilon^2+12\varepsilon-2}{\varepsilon} & 5 & & \\
s^1 : & \dfrac{\left(\dfrac{-4\varepsilon^2+12\varepsilon-2}{\varepsilon}\right)\left(\dfrac{4\varepsilon-10}{\varepsilon}\right) - 5\left(\dfrac{2\varepsilon-2}{\varepsilon}\right)}{\dfrac{-4\varepsilon^2+12\varepsilon-2}{\varepsilon}} & 0. & & \\
s^0 : & 5. & & &
\end{array}
$$

Now, apply $\varepsilon \to 0$. then the array will be.

$$
\begin{array}{lllll}
s^6 : & 1 & 1 & 3 & 5 \\
s^5 : & 2 & 2 & 4 & \\
s^4 : & 0 & 1 & 5 & \\
s^3 : & -\infty & -\infty & & \\
s^2 : & -\infty & 5 & & \\
s^1 : & -\infty & 0 & & \\
s^0 : & 5 & & &
\end{array}
$$

from the above array we conclude that there are two sign changes. So, the system is unstable, two roots lies in the right half of s-plane and remaining roots lies on the left half of s-plane.

3. $s^5+s^4+2s^3+2s^2+3s+5=0$.

Sol:

$$\begin{array}{llll} s^5 & : 1 & 2 & 3 \\ s^4 & : 1 & 2 & 5 \\ s^3 & : 0 & -2 & \end{array}$$

Replace $0 \rightarrow \varepsilon$ we get,

$$\begin{array}{llll} s^5 & : 1 & 2 & 3 \\ s^4 & : 1 & 2 & 5 \\ s^3 & : \varepsilon & -2 & \\ s^2 & : \dfrac{2\varepsilon+2}{\varepsilon} & 5 & \\ s^1 & : \dfrac{-(5\varepsilon^2+4\varepsilon+4)}{2\varepsilon+2} & & \\ s^0 & : 5. & & \end{array}$$

Now, Replace $\varepsilon \rightarrow 0$ we get,

$$\begin{array}{llll} s^5 & : 1 & 2 & 3 \\ s^4 & : 1 & 2 & 5 \\ s^3 & : 0 & -2 & \\ s^2 & : \infty & 5 & \\ s^1 & : -2 & & \\ s^0 & : 5 & & \end{array}$$

There are two sign changes in Routh Array after Replacing ϵ by 0. So, the system is unstable. Therefore, two Roots are lying on the right half of s-plane and remaining will lies on the left half of s-plane.

Alternative Method :-

Transform the characteristic equation in the s-plane to z-plane by placing s by $1/z$ and applying Routh's test. The C.E in z-plane is,

$$\left(\frac{1}{z}\right)^5 + \left(\frac{1}{4}\right)^4 + 2(1/z)^3 + 3(1/z) + 2(1/z)^2 + 5 = 0.$$

$$\Rightarrow \quad 5z^5 + 3z^4 + 2z^3 + 2z^2 + z + 1 = 0.$$

i.e

z^5	:	5	2	1
z^4	:	3	2	1
z^3	:	-1.33	-0.66	
z^2	:	0.5	1	
z^1	:	2		
z^0	:	1		

from this Method also we observed two sign changes. Therefore, the System is unstable. and the two roots are lying on the Right half of s-plane and the other roots are lying on the left half of s-plane.

Under Case - II : problems :-

① The characteristic equation of a System is $s^7+9s^6+24s^5+24s^4+24s^2+23s+15+24s^3=0$

Sol: The characteristic equation is, $s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15=0$.

The given characteristic polynomial is 7th order equation and so it has 7 roots since the highest power of 's' is odd number, form the first row of array using the coefficients of odd powers of 's' and the second row using the coefficients of even powers of 's' as shown below,

s^7 : 1 24 24 23

s^6 : 9 24 24 15

Divide the s^6 row with 3.

s^5 : $\frac{3\times24-8\times1}{3}$, $\frac{3\times24-8\times1}{3}$, $\frac{3\times23-5\times1}{3}$

s^5 : 21.33 21.33 21.33

s^5 : 1 1 1

s^4 : $\frac{1\times8-1\times3}{1}$, $\frac{1\times8-1\times3}{1}$, $\frac{1\times5-0\times3}{1}$

s^4 = 5 5 5

s^4 : 1 1 1

s^3 : $\frac{1\times1-1\times1}{1}$, $\frac{1\times1-1\times1}{1}$

: 0 0.

Auxillary polynomial $A(s) = s^4+s^2+1$

Differentiate w.r.t. s.

$\frac{dA}{ds} = 4s^3+2s$

$= 2s^3+s$

s^3 : 2 1

s^2 : $\frac{2\times1-1\times1}{2}$, $\frac{2\times1-1\times0}{2}$

: $1/2 = 0.5$, 1

$\Rightarrow$

s^7	:	1	24	24	23
s^6	:	3	8	8	5
s^5	:	1	1	1	
s^4	:	1	1	1	
s^3	:	0	0		
s^3	:	2	1		
s^2	:	0.5	1		
s^1	:	-3			
s^0	:	1			

on examining the first column element of Routh array, it is found that there are two sign changes. Hence, the two roots are lying on the Right half of s-plane and so, the system is unstable

Now, The Row of all zeros indicate the possibility of roots on imaginary axis. This can be tested by Evaluating the roots of Auxillary polynomial.

The auxillary polynomial $= s^4+s^2+1$

$$\therefore s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ} = \pm\sqrt{1}\,\angle\frac{120^\circ}{2} = \pm 1\angle 60^\circ$$

Now, put $s^2 = x$.

$$\Rightarrow x^2+x+1=0.$$

$$\Rightarrow x = \frac{-1\pm\sqrt{1-4}}{2}$$

$$= \frac{-1\pm\sqrt{-3}}{2} = \frac{-1}{2}\pm j\frac{\sqrt{3}}{2} = 1\angle 120^\circ \text{ (or) } 1\angle -120^\circ$$

$$s = \sqrt{x} = \pm(0.5 \pm j0.866) \text{ (or) } \pm(0.5 - j0.866)$$

$\therefore$ The system is unstable. As there are two sign changes therefore, two roots are lying on the right half of s-plane and Remaining are lies on the left half of splane

Alternate Method :-

s^7 : 1 24 24 23

s^6 : 3 8 8 5

s^5 : 1 1 1

s^4 : 1 1 1

s^3 : 0 0

The Auxillary polynomial $A(s) = s^4+s^2+1 = 0.$

Now, Divide the characteristic Equation with Auxillary polynomial we get a quotient polynomial and continue the Routh Array

With the quotient Polynomial.

$\therefore$ The Auxillary Polynomial $A(s) \Rightarrow s^4+s^2+1=0.$

quotient Polynomial is,

$s^3+9s^2+23s+15=0.$

$$\begin{array}{l|l}
\Rightarrow s^3 : & 1 \quad 23 \\
s^2 : & 9 \quad 15 \\
s^2 : & 3 \quad 5 \\
s^1 : & 21.33 \\
s^0 : & 5.
\end{array}$$

$$\begin{array}{r|l}
 & s^3+9s^2+23s+15 \\ \hline
s^4+s^2+1 & s^7+9s^6+24s^5+24s^4+24s^3+24s^2+23s+15 \\
 & \underline{-s^7 \quad -s^5 \quad -s^3} \\
 & 9s^6+23s^5+24s^4+23s^3+24s^2+23s+15 \\
 & \underline{-9s^6 \quad -9s^4 \quad -9s^2} \\
 & 23s^5+15s^4+23s^3+15s^2+23s+15 \\
 & \underline{-23s^5 \quad -23s^3 \quad -23s} \\
 & 15s^4+15s^2+15 \\
 & \underline{15s^4+15s^2+15} \\
 & 0
\end{array}$$

$$s^1 = \frac{3\times 23 - 5\times 1}{3} = 21.33$$

The elements of column-1 of quotient polynomial are all positive and there is no sign change. Hence all the roots of quotient polynomial are lying on the left half of s-plane. To determine the stability, the roots of auxillary polynomial should be calculated.

The Auxillary Equation $A(s) = s^4+s^2+1=0.$

Put $s^2=x.$

$$\Rightarrow x^2+x+1=0. \Rightarrow x = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$$= \frac{-1\pm\sqrt{1-4}}{2}$$

$$= \frac{-1\pm j\sqrt{3}}{2} = \frac{-1+j\sqrt{3}}{2} \text{ (or) } \frac{-1-j\sqrt{3}}{2}$$

$$= 1\angle 120^\circ \text{ (or) } 1\angle -120$$

$$\Rightarrow s = \pm\sqrt{x} = \pm\sqrt{1\angle 120^\circ}, \pm\sqrt{1\angle -120}$$

$$= \pm\sqrt{1}\sqrt{\angle 120}, \pm\sqrt{1}\sqrt{\angle -120}$$

$$= \pm 1\angle 120/2, \pm 1\angle -120/2$$

$$= \pm 1\angle 60, \pm 1\angle -60$$

$= \pm (0.5 + j0.866)$, $\pm (0.5 - j0.866)$.

The roots of auxillary equation are complex and has quadrantal symmetry. Two roots of auxillary equation are lying on the right half of s-plane and the other two on the left half of s-plane.

∴ The roots of C.E are given by the roots of auxillary equation and the quotient equation. Hence, the two roots are lying on the right half of s-plane. Therefore, the system is unstable. The Remaining roots are lying on the left half of s-plane.

Q. Comment on stability of the system with the following C.E.

$$D(s) = s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25.$$

Sol. given, characteristic equation

$$D(s) = s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25 = 0$$

The given characteristic equation is of the order 6. So, it has 6 roots. Since, the highest power of 's' is even number so, the Routh Array starts with the even power coefficients in the first row, then and the second row using the coefficients of odd powers of 's' as shown below.

s^6 :	1	7	31	25
s^5 :	1	6	25	
s^4 :	1	6	25	
s^3 :	0	0		

→ This coefficients of powers will forms a Auxillary polynomial.

∴ The auxillary polynomial $A(s) = s^4 + 6s^2 + 25$.

$$\frac{d\,A(s)}{ds} = 4s^3 + 12s.$$

Now, Substitute the Coefficients of the differential Auxillary Polynomial, we gets

s^6 :	1	7	31	25
s^5 :	1	6	25	
s^4 :	1	6	25	
s^3 :	4	12		
s^2 :	3.	25		
s^1 :	$\frac{-64}{3}$			
s^0 :	25			

$$s^1 = \frac{3\times 12 - 4\times 25}{3} = \frac{36-100}{3} = \frac{-64}{3}.$$

Here, the two sign changes are observed in first Column. So, the two roots are lying on the right half of s-plane. and remaining roots are lying on left-half of s-plane. Thus, the System is unstable.

As the Row of Zeros indicates, there may exists a purely imaginary, purely real (or) purely Complex. roots. These will get with the help of Auxillary polynomial.

$$\therefore \quad s^4 + 6s^2 + 25 = 0.$$

$$\Rightarrow \text{Put } s^2 = x. \quad \Rightarrow \quad x^2 + 6x + 25 = 0.$$

$$x = \frac{-6 \pm \sqrt{36 - 25\times 4}}{2} = \frac{-6 \pm \sqrt{-64}}{2} = \frac{-6 \pm j8}{2}$$

$$= -3 \pm j4.$$

$$= 5\angle 126.86,\ 5\angle -126.86$$

$$\Rightarrow s^2 = x \Rightarrow s = \sqrt{x} = \pm\sqrt{x} = \sqrt{5\angle 126.86},\ \sqrt{5\angle -126.86}$$

$= \pm\left(\sqrt{5}\ \angle\frac{126.86}{2}\right),\ \pm\left(\sqrt{5}\ \angle\frac{-126.86}{2}\right)$

$= \pm(2.23\ \angle 63.43),\ \pm(2.23\ \angle -63.43)$

$= \pm(1 \pm j2),\ \pm(1 \mp j2).$

$= \pm(1+j2),\ \pm(1-j2).$

~~Now,~~ Alternative Method :-

s^6	:	1	7	31	25
s^5	:	1	6	25	
s^4	:	1	6	25	
s^3	:	0	0.		

Auxillary polynomial $A(s) = s^4 + 6s^2 + 25 = 0.$

Now, Divide the characteristic polynomial with Auxillary polynomial. we will get one quotient polynomial. Then Construct the Routh array for the quotient polynomial.

$$
\begin{array}{r|l}
 & s^2 + s + 1 \\
\hline
s^4+6s^2+25 & s^6 + s^5 + 7s^4 + 6s^3 + 31s^2 + 25s + 25. \\
 & \underline{s^6 \quad\quad 6s^4 \quad\quad\quad 25s^2 \quad \cancel{25s}} \\
 & s^5 + s^4 + 6s^3 + 6s^2 + 25s + 25 \\
 & \underline{s^5 \quad\quad 6s^3 \quad 6s^2 \quad 25s} \\
 & s^4 + 6s^2 + 25 \\
 & \underline{s^4 + 6s^2 + 25} \\
 & 0
\end{array}
$$

$\therefore$ The quotient polynomial $q(s) = s^2 + s + 1$. so.

s^2	:	1	1
s^1	:	1	
s^0	:	1	

$s^0 = \frac{1\times1 - 1\times0}{1}$

$= 1$

As there are no sign changes in the quotient polynomial, Due to the presence of a row of zeros, there may exists an imaginary roots. This will get from the Auxillary Polynomial,

$A(s) = s^4 + 6s^2 + 25 = 0.$

$\Rightarrow$ put $s^2 = x.$

$\Rightarrow x^2 + 6x + 25 = 0.$

$\Rightarrow x = \dfrac{-6 \pm \sqrt{36-100}}{2} = \dfrac{-6 \pm j8}{2}$

$= -3 \pm j4, \ -3-j4$

$= 5\angle 126.86, \ 5\angle -126.86$

Now, $s^2 = x \Rightarrow s = \sqrt{x} = \pm\sqrt{x} = \sqrt{5\angle 126.86}, \sqrt{5\angle -126.86}$

$= \pm\sqrt{5}\angle \frac{126.86}{2}, \ \pm\sqrt{5}\angle \frac{-126.86}{2}$

$= \pm(2.23\angle 63.43), \ \pm(2.23\angle -63.43)$

$= \pm(1+j2), \ \pm(1-j2).$

Now the number of roots on right half of s-plane will be decided that the number of positive Real part roots in obtained from Auxillary. Polynomial.

So, As there are two positive Real part root so, the two roots are lies on right half of s-plane and the remaining roots lies on left half of s-plane.

Therefore, the system is unstable

3. $s^6+2s^5+8s^4+12s^3+20s^2+16s+16$.

Sol. given characteristic equation,

$$\Rightarrow s^6+2s^5+8s^4+12s^3+20s^2+16s+16.$$

s^6 :	1	8	20	16
s^5 :	2	12	16	
s^4 :	2	12	16	→ Auxillary polynomial A(s).
s^3 :	0	0	0	

$$\therefore A(s) = 2s^4+12s^2+16.$$
$$= s^4+6s^2+8.$$

$$\Rightarrow A(s) = s^4+6s^2+8$$

$$\frac{d\,A(s)}{ds} = 4s^3+12s.$$

s^6 :	1	8	20	16
s^5 :	2	12	16	
s^4 :	2	12	16	
s^3 :	4	12		
s^2 :	3	8		
s^1 :	0.33			
s^0 :	8			

Here, all the elements in the first column are positive. Also, we got a Row of zeros. So, the system is marginally (or) limitedly stable. The Roots of Auxillary polynomial are,

$$A(s) \Rightarrow s^4+6s^2+8=0$$

Put, $s^2=x \Rightarrow x^2+6x+8=0$.

$$\Rightarrow x = \frac{-6 \pm \sqrt{36-32}}{2} = \frac{-6 \pm \sqrt{4}}{2}$$

$$= \frac{-6 \pm 2}{2} \Rightarrow \frac{-6+2}{2}, \quad \frac{-6-2}{2}$$

$$\Rightarrow -2, \quad -4$$

$$\therefore \quad s = \pm\sqrt{x} = \pm\sqrt{-2}, \quad \pm\sqrt{-4}$$

$$= \pm j\sqrt{2}, \quad \pm j\sqrt{4}$$

$$= \pm j\sqrt{2}, \quad \pm j2$$

$$= \pm j1.414, \quad \pm j2$$

As we got the two distinct purely imaginary roots. So, the system is Marginally (or) Limitedly stable.

Alternate Method :-

given characteristic equation $\Rightarrow s^6+2s^5+8s^4+12s^3+20s^2+16s+16=0$

Now,

s^6 :	1	8	20	16
s^5 :	1	6	8	
s^4 :	1	6	8	
s^3 :	0	0	0.	

$\therefore$ The Auxillary polynomial $A(s) \Rightarrow s^4+6s^2+8=0.$

Divide the main characteristic equation with Auxillary equation to obtain the quotient polynomial.

s^4+6s^2+8) $s^6+2s^5+8s^4+12s^3+20s^2+16s+16$ ($s^2+2s+2.$

$\quad \cancel{s^6} \quad 6s^4 \quad 8s^2$

$\quad 2s^5+2s^4+12s^3+12s^2+16s+16$

$\quad \cancel{2s^5} \quad \cancel{12s^3} \quad \cancel{16s}$

$\quad 2s^4+12s^2+16$

$\quad \cancel{2s^4}+\cancel{12s^2}+\cancel{16}$

$\quad 0$

$\therefore$ The quotient polynomial $q(s) = s^2+2s+2$

$$\Rightarrow \begin{array}{lll} s^2 : & 1 & 2 \\ s^1 : & 2 & \\ s^0 : & 2 & \end{array}$$

In the first column of quotient polynomial also there are no sign changes. So, to know (or) determine the system is stable (or) unstable we have to find the roots for the Auxillary polynomial.

$\therefore \quad A(s) \Rightarrow s^4+6s^2+8=0$

Put $s^2 = x$

$\Rightarrow x^2+6x+8=0.$

$\Rightarrow x = \dfrac{-6 \pm \sqrt{36-32}}{2}$

$= \dfrac{-6 \pm \sqrt{4}}{2}$

$= \dfrac{-6 \pm 2}{2} \Rightarrow \dfrac{-6+2}{2}, \quad \dfrac{-6-2}{2}$

$\Rightarrow -2, \quad -4.$

$\therefore \quad s = \pm\sqrt{x}$

$= \pm\sqrt{-2}, \quad \pm\sqrt{-4}$

$= \pm j\sqrt{2}, \quad \pm j\sqrt{4}$

$= \pm j\sqrt{2}, \quad \pm j2$

$= \pm j1.414, \quad \pm j2$

Here, there are two imaginary roots on both the sides of imaginary axis. And these are distinct roots. So, the system is marginally stable (or) limitedly stable

4. Determine the range of K for stability of unity feedback system whose open loop transfer function is

$$G(s) = \frac{K}{s(s+1)(s+2)}.$$

Sol: given open loop T.F $G(s) = \frac{K}{s(s+1)(s+2)}$.

We know that, the closed loop Transfer function is,

$$T.F = \frac{G(s)}{1+G(s)H(s)} \quad [\because H(s)=1]$$

$$\Rightarrow T.F = \frac{\frac{K}{s(s+1)(s+2)}}{1+\frac{K}{s(s+1)(s+2)}} = \frac{K}{s(s+1)(s+2)+K}.$$

Now, the characteristic equation is $1+G(s)H(s)=0$.

$$\Rightarrow \text{i.e } s(s+1)(s+2)+K=0.$$

$$\Rightarrow s(s^2+3s+2)+K=0$$

$$\Rightarrow s^3+3s^2+2s+K=0.$$

Now, the Routh Array is constructed as follows,

$$\begin{array}{lcll} s^3 & : & 1 & 2 \\ s^2 & : & 3 & K \\ s^1 & : & \frac{6-K}{3} & \\ s^0 & : & K. & \end{array}$$

For, the system to be stable there should not be any sign change in the elements of first column. Hence, choose the value of K so that the first column elements are positive from s^0 Row, for the system to be stable, $K > 0$.

from s^1 row, for the system to be stable, $\frac{6-K}{3} > 0$.

$\Rightarrow 6-K > 0$

$\Rightarrow 6 > K \Rightarrow K < 6$.

$\therefore$ The value of 'K' should be in the range of $0 < K < 6$ for the system to be stable.

5. open loop transfer function with unity feed back of a system is given by $G(S) = \frac{K}{(S+2)(S+4)(S^2+6S+25)}$. Determine the value of K which will cause sustained oscillation in the closed loop system.

Sol: The closed loop T.F $= \frac{G(S)}{1+G(S)} = \frac{\frac{K}{(S+2)(S+4)(S^2+6S+25)}}{1+\frac{K}{(S+2)(S+4)(S^2+6S+25)}}$

$$= \frac{K}{(S+2)(S+4)(S^2+6S+25)+K}$$

Now, the characteristic equation is,

$$(S+2)(S+4)(S^2+6S+25)+K = 0.$$

$$\Rightarrow (S^2+6S+8)(S^2+6S+25)+K=0.$$

$$\Rightarrow S^4+12S^3+69S^2+198S+200+K=0.$$

$\therefore$

S^4 :	1	69	$200+K$
S^3 :	$\frac{12}{1}$	$\frac{198}{16.5}$
S^2 :	52.5	$200+K$.
S^1 :	$\frac{666.25-K}{52.5}$	
S^0 :	$200+K$.	

S^1: $\frac{52.5\times16.5-200+K}{52.5}$

$= \frac{666.25-K}{52.5}$

For the System to be stable there should be no sign changes in first Coloumn and Choose the value of K so, that the first elements of first Coloumn are positive

$\therefore$ from s^1 row $\Rightarrow$ $666.25 - K > 0$

$\Rightarrow$ $K < 666.25$

from s^0 row, $\Rightarrow$ $200+K>0 \Rightarrow 200+K > 0$

$\Rightarrow -K < +200$

$\Rightarrow K > -200$

$\therefore$ The Range of K for the System to be stable is,

$0 < K < 666.25$.

Now, the System will oscillates when $K = 666.25$ then the s^1 row becomes zero, which indicates the possibility of imaginary roots on s-plane. A System will oscillates when the roots are on imaginary axis and no root is on Right half of s-plane.

when $K = 666.25$, then the auxillary equation is,

$$A(s) = 52.5s^2 + (200+K) = 0.$$

$$\Rightarrow 52.5s^2 + 200 + 666.25 = 0$$

$$\Rightarrow 52.5s^2 + 866.25 = 0$$

$$\Rightarrow s^2 = \frac{-866.25}{52.5} = -16.5$$

$$\Rightarrow s = \pm\sqrt{-16.5}$$

$$= \pm j\underset{\omega}{4.06} \quad \text{Similar to } \pm j\omega.$$

$\therefore$ when $K = 666.25$ the System will oscillates and the frequency of oscillation $\omega = 4.06$ rad/sec

6. A unity feedback control system is characterized by open-loop transfer function. $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$.

i. Calculate the range of values of K for the system to be stable

ii, what is the marginal value of K for stability? Determine the frequency of oscillations.

The characteristic equation is $1 + G(s)H(s) = 0$.

$$\Rightarrow 1 + \frac{K(s+13)}{s(s+3)(s+7)} = 0$$

$$\Rightarrow s(s+3)(s+7) + K(s+13) = 0.$$

$$\Rightarrow s^3 + 10s^2 + (21+K)s + 13K = 0.$$

∴ Routh Array is tabulated as,

$$\begin{array}{lll} s^3 : & 1 & 21+K \\ s^2 : & 10 & 13K \\ s^1 : & \frac{10(21+K) - 13K}{10} = \frac{210 - 3K}{10} & \\ s^0 : & 13K. & \end{array}$$

i, For the system to be stable, the first column elements of Routh table is should be positive.

from the s^0 row, $13K > 0$

$\Rightarrow K > 0$.

from the s^1 row, $210 - 3K > 0$.

$\Rightarrow 3K < 210$

$\Rightarrow K < 70$

∴ Hence, the range of values of K for stability is $0 < K < 70$.

(ii). The System will oscillate for $K = 70$, then the s^1 row will becomes zero, which indicates the possibility of imaginary roots.

When $K = 70$ the Auxillary equation is,

$$A(s) \Rightarrow 10s^2 + 13K = 0$$

$$\Rightarrow 10s^2 + 13 \times 70 = 0 \quad (\because K = 70)$$

$$\Rightarrow 10s^2 = \cancel{210} -13 \times 70$$

$$\Rightarrow s^2 = \frac{\cancel{21} -13 \times 70}{10}$$

$$\Rightarrow s^2 = \sqrt{\frac{-13 \times 70}{10}} = \pm j\sqrt{\frac{13 \times 70}{10}}$$

$$= \pm j\, 9.53.$$

$\therefore$ The frequency of oscillation is $\omega = 9.53$ rad/sec

(2) $2s^5+2s^4+5s^3+5s^2+3s+5=0$ 1

B.N. Prasad

Sol:

$s^5 : 2 \quad 5 \quad 3$

$s^4 : 2 \quad 5 \quad 5$

$s^3 : 0 \quad -2$

$s^3 : \frac{2\times5-2\times5}{2}=0$;

$\frac{2\times3-2\times5}{2} = \frac{6-10}{2} = \frac{-4}{2}$.

Now Replace an Multiple integral ε in place of zero.

$\therefore \quad s^3 : \varepsilon \quad -2$

$s^2 : \frac{5\varepsilon+4}{\varepsilon} \quad \varepsilon$

$s^1 : \frac{-2\left(\frac{5\varepsilon+4}{\varepsilon}\right)-\varepsilon^2}{\frac{5\varepsilon+4}{\varepsilon}}$

$s^0 : \varepsilon$

$s^4 : 2 \quad 5 \quad 5$

$s^3 : \varepsilon \quad -2$

$s^2 : \frac{\varepsilon 5+4}{\varepsilon} \quad \frac{5\varepsilon}{\varepsilon}$

$s^1 : -2$

Now Replace 0 in Place of multiple integral ε

$\Rightarrow \quad s^5 : 2 \quad 5 \quad 3$

$s^4 : 2 \quad 5 \quad 5$

$s^3 : 0 \quad -2$

$s^2 : \infty \quad 0$

$s^1 : -\infty$

$s^0 : 0$

$\therefore$ Two sign changes, so, the two Roots lies in the Right half of s-plane and Remaining roots in L.H. s-plane

$\therefore$ System is unstable

(b) $2s^6+4s^5+s^4-32s^3+51s^2+3s+15=0$

Sol

s^6 : 2 1 51 15

s^5 : 4 -32 3

s^4 : 17 49.5 15

s^3 : -43.64 -0.52

s^2 : 49.29 15

s^1 : 12.76

s^0 : 15

s^4 : $\frac{4+64}{4} = \frac{68}{4} = 17$

$\frac{4\times51-6}{4} = 49.5$

s^3 : $\frac{17\times-32-4\times49.5}{17}$

$\frac{-544-198}{17} = -43.64$

$\frac{17\times3-4\times15}{17}$

s^2 : $\frac{-43.64\times49.5+17\times0.52}{-43.64}$

$\frac{-2160.18+8.84}{-43.64}$

s^1 : $\frac{+43.64\times15+49.29\times0.52}{49.29}$

$\frac{654.6-25.63}{49.29}$

12.76

s^0 : $\frac{12.76\times15-0}{12.76}$

$= 15$

∴ As there is two sign changes. So, two roots lies in right half of s-plane and Remaining roots lies in left half of s-plane.

∴ The System is Unstable

(c) $s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$

Sol

s^6 : 1 5 8 4

s^5 : 3 9 6

s^4 : 2 6 4

s^3 : 0 0

s^4 : $\frac{3\times5-9}{3}$; $\frac{3\times8-1\times6}{3}$; $\frac{3\times4-0}{3}$

2 6 4

s^3 : $\frac{2\times9-3\times6}{2}$; $\frac{2\times6-3\times4}{2}$

0 ; 0

A Row of all zeros.

Now, the Auxillary polynomial $A(s) = 2s^4 + 6s^2 + 4$

$$\therefore \frac{dA(s)}{ds} = 8s^3 + 12s$$

$$\therefore \begin{array}{llll} s^4 : & 2 & 6 & 4 \\ s^3 : & 8 & 12 & \\ s^2 : & 3 & 4 & \\ s^1 : & 1.33 & & \\ s^0 : & 4 & & \end{array}$$

$s^2 : \frac{8\times6 - 2\times12}{8} \; ; \; \frac{8\times4 - 0}{8}$

$3 \quad : \quad 4$

$s^1 : \frac{3\times12 - 8\times4}{3}$

$\frac{36-32}{3} = \frac{4}{3}$

$s^0 : \frac{\cancel{1.33}\times4}{\cancel{1.33}}$

As there is no sign changes, even though we can't judge the stability of a system

As the Row becomes zero, there is a possibility of purely imaginary, purely Real (or) Complex roots. This can be obtained by Auxillary polynomial.

$$2s^4 + 6s^2 + 4 = 0$$

let $s^2 = x$ $\therefore 2x^2 + 6x + 4 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm \sqrt{36 - 4\times2\times4}}{2\times2}$$

$$= \frac{-6 \pm \sqrt{36-32}}{4} = \frac{-6 \pm \sqrt{4}}{4}$$

$$= \frac{-6 \pm 2}{4} = \frac{-6+2}{4} \; ; \; \frac{-6-2}{4}$$

$$= \frac{-4}{4} \; ; \; \frac{-8}{4}$$

$$\therefore s^2 = -1, -2 \Rightarrow s = \sqrt{-1}, \sqrt{-2} = \pm j1, \pm j2 = -1, -2$$

(d) Alternate method for $2s^5 + 2s^4 + 5s^3 + 5s^2 + 3s + 5 = 0$

Sol: Substitute $s = \frac{1}{z}$ in given characteristic equation.

we get, $2\left(\frac{1}{z}\right)^5 + 2\left(\frac{1}{z}\right)^4 + 5\left(\frac{1}{z}\right)^3 + 5\left(\frac{1}{z}\right)^2 + 3 \times \frac{1}{z} + 5 = 0$

$\Rightarrow 5z^5 + 3z^4 + 5z^3 + 5z^2 + 2z + 2 = 0$

Now, Routh-Array,

z^5 :	5	5	2
z^4 :	3	5	2
z^3 :	−3.33	−1.33	
z^2 :	0.801	2	
z^1 :	6.98		
z^0 :	2		

$\frac{3\times5 - 5\times5}{3}$; $\frac{3\times2 - 5\times2}{3}$

-3.33 ; $\frac{6-10}{3}$

z^2 : $\frac{-3.33\times2 + 1.33\times3}{-3.33}$; $\frac{-3.33\times2 - 0}{-3.33}$

z^1 : $\frac{0.801\times -1.33 + 3.33\times2}{0.801}$

$= \frac{-1.065 + 6.66}{0.801}$

$= 6.98$

z^0 : $\frac{6.98 \times 2 - 0}{6.98}$

$= 2$

There are two sign changes, so, two roots are lying in right half of s-plane and remaining roots are lying in left half of s-plane

∴ The System is unstable

(e) Alternate Method for $s^6+3s^5+5s^4+9s^3+8s^2+6s+4=0$ (2)

Sol.

$s^6 : 1 \quad 5 \quad 8 \quad 4$

$s^5 : 3 \quad 9 \quad 6$

$s^4 : 2 \quad 6 \quad 4 \Rightarrow 1 \quad 3 \quad 2$

$s^3 : \underline{\underline{0 \quad 0}}$

$\frac{15-9}{3} = \frac{6}{3}$; $\frac{3\times 8 - 1\times 6}{3}$; $\frac{\cancel{3}\times 4 -}{\cancel{3}}$

Now, the Auxillary polynomial $A(s) = 2s^4+3s^2+2=0$.

Now, Divide the given characteristic equation with Auxillary equation

$2s^4+3s^2+2 \;\big)\; s^6+3s^5+5s^4+9s^3+8s^2+6s+4 \;\big(\; s^2+3s+2$

$\quad (-)\,2s^6 \quad (-)\,3s^4 \quad (-)\,2s^2$

$\quad 3s^5+2s^4+9s^3+6s^2+6s+4$

$\quad (-)\,3s^5 \quad (-)\,9s^3 \quad (-)\,6s$

$\quad 2s^4+6s^2+4$

$\quad (-)\,2s^4+6s^2+4$

$\quad (0)$.

Now, quotient polynomial $Q(t) = s^2+3s+2$

$s^2 : 1 \quad 2$

$s^1 : 3$

$s^0 :$

As there is no sign changes in first column of Routh-Array of quotient polynomial and we have got a Row of zeros. So, the system will be limited by stable

From the Auxillary polynomial the purely imaginary Roots will be find out

$$A(s) = s^4 + 3s^2 + 2 = 0$$

$\Rightarrow$ let, $s^2 = x \Rightarrow x^2 + 3x + 2 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{9 - 2 \times 4 \times 1}}{2}$$

$$= \frac{-3 \pm \sqrt{9-8}}{2}$$

$$= \frac{-3 \pm \sqrt{1}}{2}$$

$$= \frac{-3 \pm 1}{2} = \frac{-3+1}{2}, \frac{-3-1}{2}$$

$$= \frac{-2}{2}, \frac{-4}{2}$$

$$= -1, -2$$

$$\therefore s = \sqrt{x} = \sqrt{-1}, \sqrt{-2}$$

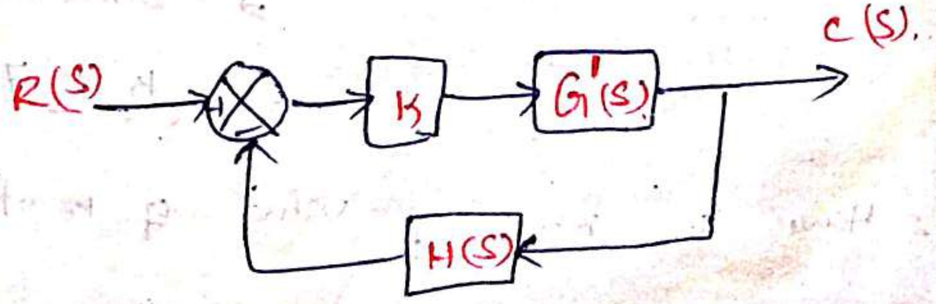
$$= \pm j1 ; \pm j2$$

* Concept of Root locus:-

1, The Root locus Technique was introduced by W.R. Evans. i.e. an American Scientist in 1948. This is a graphical Method, in which movement of Poles in the S-plane is sketched when a particular parameter of a system is varied from "zero to infinity". Note that the parameter is usually the gain but any other parameter may be varied. But for Root locus Method, "gain" is assumed to be a parameter which is to be varied from zero to infinity.

* Basic Concept of Root locus:-

In general, the characteristic equation of a closed loop system is given by,



$$\boxed{1+G(S)H(S)=0}$$

For the Root locus, the gain K is assumed to be a variable parameter and is a part of forward path of the closed loop System.

$$\therefore \quad G(s) = K \cdot G'(s).$$

Where K = Gain of the amplifier in the forward path or also called System Gain.

Now, the characteristic Equation becomes.

$$1 + G(s)H(s) = 0 \quad \text{i.e} \quad 1 + K \cdot G'(s)H(s) = 0.$$

which Contains 'K' as a variable parameter

Now, if gain K is varied from $-\infty$ to ∞ then for Each
Seperate value of 'K' we will get seperate set of locations
of the roots for the characteristic equation. If all such
locations are joined the resulting locus is called Root locus.

Definition:-

The Root locus can be defined as, the locus of closed
loop poles obtained when a System gain 'K' is varied from
0 to $-\infty$ is called Root locus.

* Angle and Magnitude Conditions:-

For a general Closed loop C.E

$$1 + G(s)H(s) = 0.$$

$$\text{i.e} \quad G(s)H(s) = -1$$

As s-plane is Complex we can write above eq.

$$\boxed{G(s)H(s) = -1 + j0.}$$

All s-values can be expressed as $\sigma + j\omega$ i.e., $G(S)H(S)$ term is also a complex one. So for any value of 's' if it has to be on the root locus, it must satisfy the above equation.

As both sides of above equations are in Rectangular form, convert both sides into polar form and then equate angle and magnitudes of both sides. This gives two condition called as

1. Angle Condition. 2. Magnitude Condition.

* Angle Condition :-

$$G(S)\,H(S) = -1+j0.$$

Equating angles of both sides,

$$\Rightarrow \angle G(S)\,H(S) = \pm(2q+1)\,180^\circ \qquad q = 0,1,2,\ldots$$

This means the angle of $G(S)H(S)$ should be an odd multiple of 180°. If this condition is met, then for any root, lies on the Root locus path.

For any root of 's' this condition should satisfy other than the root does not belongs to Root locus.

$\therefore$ Angle Condition can be stated as,

$|\angle G(S)\,H(S)|$ for any value of 's' which is the root of equation $1+G(S)H(S)=0$ is,

$$= \pm(2q+1)\,180^\circ \qquad \text{where, } q = 0,1,2,3,\ldots$$

$$= \text{odd Multiple of } 180^\circ.$$

Eg: Consider a system with $G(S)H(S) = \frac{K}{s(s+2)(s+4)}$. Find whether the root of $s = -0.75$ is on the root locus (or) not using angle condition.

Sol: Angle condition $\Rightarrow$ $\angle G(s)H(s) = \pm(2q+1)180°$.

Now, substituting the value $s = -0.75$

$$\Rightarrow \angle G(s)H(s)\Big|_{s=-0.75} = \frac{\angle K+j0}{\angle(-0.75+j0)\angle(1.25+j0)\angle(3.25+j0)}$$

$$= \frac{0°}{180°\cdot 0°\cdot 0°} = -180°.$$

which is the odd multiple of 180°. Therefore, the root of characteristic equation $s = -0.75$ is lies on the root locus.

Now, check for $s = -1+j4$.

$$\angle G(s)H(s)\Big|_{s=-1+j4} = \frac{\angle K+j0}{\angle(-1+j4)\angle(1+j4)\angle(3+j4)}$$

$$= \frac{0°}{(104.33°)\cdot(75.963°)\cdot(53.13°)}$$

$$= -233.123°$$

which is not the odd multiple of 180°. Therefore, the root $s = -1+j4$ is not lies on the root locus.

* Magnitude condition :-

If magnitudes of both sides of the equation $G(s)H(s) = -1$ are equated then we get a magnitude condition.

$$|G(s)H(s)| = |-1+j0| = |-1| = 1$$

$$\therefore \boxed{|G(s)H(s)| = 1}$$

The Magnitude Condition is not suitable to check the existance of a particular root on the Root locus. But once we know a point in s-plane is on the Root locus using angle condition, then it must satisfy the magnitude condition also.

So, at a particular point which is known to be on Root locus using angle condition, we can find a value of 'K' by magnitude condition. The 'K' is the gain of the characteristic equation.

$\therefore$ Magnitude condition is, $\boxed{|G(s)\,H(s)| = 1.}$

Eg: Refer to previous Example $G(s)\,H(s) = \dfrac{K}{s(s+2)(s+4)}$ and $s = -0.75$ is confirmed that lies on Root locus using angle condition. Now, find value of K using magnitude condition.

Sol.

$$\left. G(s)\,H(s) \right|_{s=-0.75} = 1$$

$$\Rightarrow \frac{|K|}{|-0.75|\,|1.25|\,|3.25|} = 1$$

$$\Rightarrow \frac{K}{(3.0468)} = 1$$

$$\Rightarrow K = 3.0468$$

for the characteristic equation $s^3 + 6s^2 + 8s + K = 0$ we decided that among the 3 roots $s = -0.75$ is one of the root of characteristic equation using Angle and Magnitude conditions.

* Rules for the Construction of Root locus:-

Rule-1:- The Root locus is symmetrical about Real axis.

Rule-2:- Each branch of root locus starts (or) originates from a pole and terminates at either a finite zero (or) a zero at infinity. The number of branches determines based on the difference of finite open loop poles to the finite number of zeros.

Rule-3:- Segments of the real axis having an odd number of poles and zeros to their right are the parts (or) branches of Root locus.

Rule-4:- To find the angle of Assymptotes which is given by,

$$\phi = \frac{\pm 180 (2q+1)}{n-m} \quad ; \quad q = 0, 1, 2, \ldots (n-m).$$

where, n = number of poles and

m = number of zeros.

Rule-5:- To find the Centroid which is a point on Real axis from which the assymptotic angles have to be drawn is given by,

$$\sigma_A = \frac{\text{Sum of no of poles} - \text{Sum of no of zeros}}{n - m}.$$

Rule-6: The breakaway and breakin points of root locus are determined from the roots of the equation $\frac{dK}{ds} = 0$.

Rule-7: The angle of departure from a Complex pole is given by $180 - \phi_1 + \phi_2$

where, ϕ_1 is the net angle Contribution to the Complex pole from other Poles and ϕ_2 is net angle Contribution to the Complex pole from the zeros.

Similarly, the angle of arrival is given by,

$180° - \phi_2 + \phi_1$.

ϕ_1 = net angle Contribution of Poles and ϕ_2 is net angle Contribution to the Complex zero from other zeros.

Rule-8: The intersection point of root locus on imaginary axis can be determined by the use of R-H Criterion (or) by letting $s = j\omega$ in Characteristic Equation and equating Real Part and imaginary Part to zero, to Solve ω and K.

And the value of ω is the intersection point on imaginary axis and K is the gain at the intersection point.

*** Procedure to Construct the Root locus:-**

Step-1: Locate the poles and zeros of $G(s).H(s)$ on the s-plane. The root locus branch Start from open loop poles and terminates at zeros.

Step -2 :- Determine the Root locus on the Real axis.

Step -3 :- Determine the assymptotes of root locus branches and meeting point of asymptotes with real axis.

Step -4 :- Find the breakaway and break in point

Step 5 :- If there is a Complex pole then determine the angle of departure from the Complex pole. If there is a Complex zero then determine the angle of arrival at complex zero.

Step -6 :- Find the points where the root loci may cross the imaginary axis.

Step -7 :- Take a series of Test points in the broad neighbourhood of the origin of the s-plane and adjust the test point to satisfy the angle criterion. Sketch the root locus by joining the test point with the help of smooth curve

Problem:-

1. Sketch the root locus of the system whose open loop transfer function is, $G(S) = \frac{K}{s(s+2)(s+4)}$.

Sol. 1. locate poles and zeros.

of equation, $s(s+2)(s+4)=0$

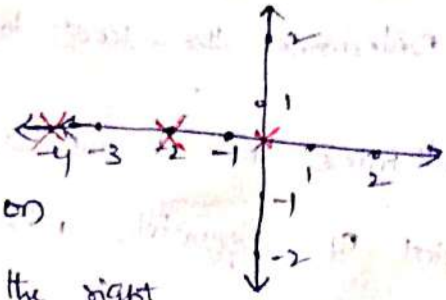
The poles of open loop transfer function is $-2, -4, 0$.

$\therefore$ number of poles $n = 3$

and zeros $= 0$. ($\because$ There is no 's' term in Numerator).

Step-2 :

There are 3 Poles on Real axis. Choose a test point on real axis between $S=0, -2$. To the right



of this point the total number of real poles and zeros is one which is odd number. Hence, the real axis between $S=0$ & $S=-2$ will be a Part of Root locus.

Similarly. $S=-4$, $S=-\infty$ there exists a 3 number of Poles which is odd number. Therefore, from $S=-4$ to $S=-\infty$ will be a part of Root locus.

3. Angle of Assymptotes $= \dfrac{\pm 180 (2q+1)}{n-m}$ $(q=0,1,2,\dots n-m)$

$$\Rightarrow q=0 \Rightarrow \frac{\pm 180 (1)}{3-0} = \pm 60^\circ$$

$$q=1 \Rightarrow \frac{\pm 180 (2+1)}{3} = \pm 180^\circ$$

$$q=2 \Rightarrow \pm \frac{180 \times 5}{3} = \pm 300 = \mp 60^\circ.$$

4. Centroid $= \dfrac{\text{Sum of Poles} - \text{Sum of zeros}}{\text{Poles} - \text{zeros}}$

$$= \frac{0-2-4-0}{3} = \frac{-6}{3} = -2.$$

5. To find Breakaway point or Breakin point

$$\text{Closed loop T.F} = \frac{G(S)}{1+G(S)} = \frac{\dfrac{K}{s(S+2)(S+4)}}{1+\dfrac{K}{s(S+2)(S+4)}}$$

$$= s(S+2)(S+4)+K=0$$

$\Rightarrow \quad s(s+2)(s+4)+K=0$

$\Rightarrow \quad K = -(s^3+6s^2+8s)$.

Now, differentiate above eq. w.r.t. s. and equate to zero.

$$\Rightarrow \frac{dK}{ds} = -(3s^2+12s+8) = 0$$

$$\Rightarrow -(3s^2+12s+8) = 0$$

$$\Rightarrow s = \frac{-12 \pm \sqrt{12^2-4\times3\times8}}{2\times3}$$

$$= -0.845 \text{ (or) } -3.154.$$

Now check for K.

when $s = -0.845$, the value of K is,

$$K = -(s^3+6s^2+8s)$$

$$= -\left[(-0.845)^3+6(-0.845)^2+(8\times-0.845)\right]$$

$$= \underline{\underline{3.08}}$$

Since, K is positive for $s = \underline{\underline{-0.845}}$ So, $s=-0.845$ is a actual Breakaway point.

when $s = -3.154$

$$K = -\left[(-3.154)^3+6(-3.154)^2+8(-3.154)\right]$$

$$= \underline{\underline{-3.08}}$$

Since K is negative therefore, $s=-3.154$ is not a Breakaway point.

6. Since, there is no Complex Pole (or) Complex zero.

So, there is no angle of Departure (or) angle of Arrival.

7. Crossing Point on Imaginary axis.

Characteristic equation is given by,

$$s^3+6s^2+8s+K=0.$$

Put $s=j\omega$

$$\Rightarrow (j\omega)^3+6(j\omega)^2+8j\omega+K=0$$

$$\Rightarrow -j\omega^3-6\omega^2+8j\omega+K=0.$$

Now, equate Real part & imaginary parts to zero.

$$\Rightarrow -j\omega^3+8j\omega=0$$

$$\Rightarrow j\omega^{3\,2}=8j\omega$$

$$\omega^2=8$$

$$\omega=\pm\sqrt{8}$$

$$=\pm 2.8$$

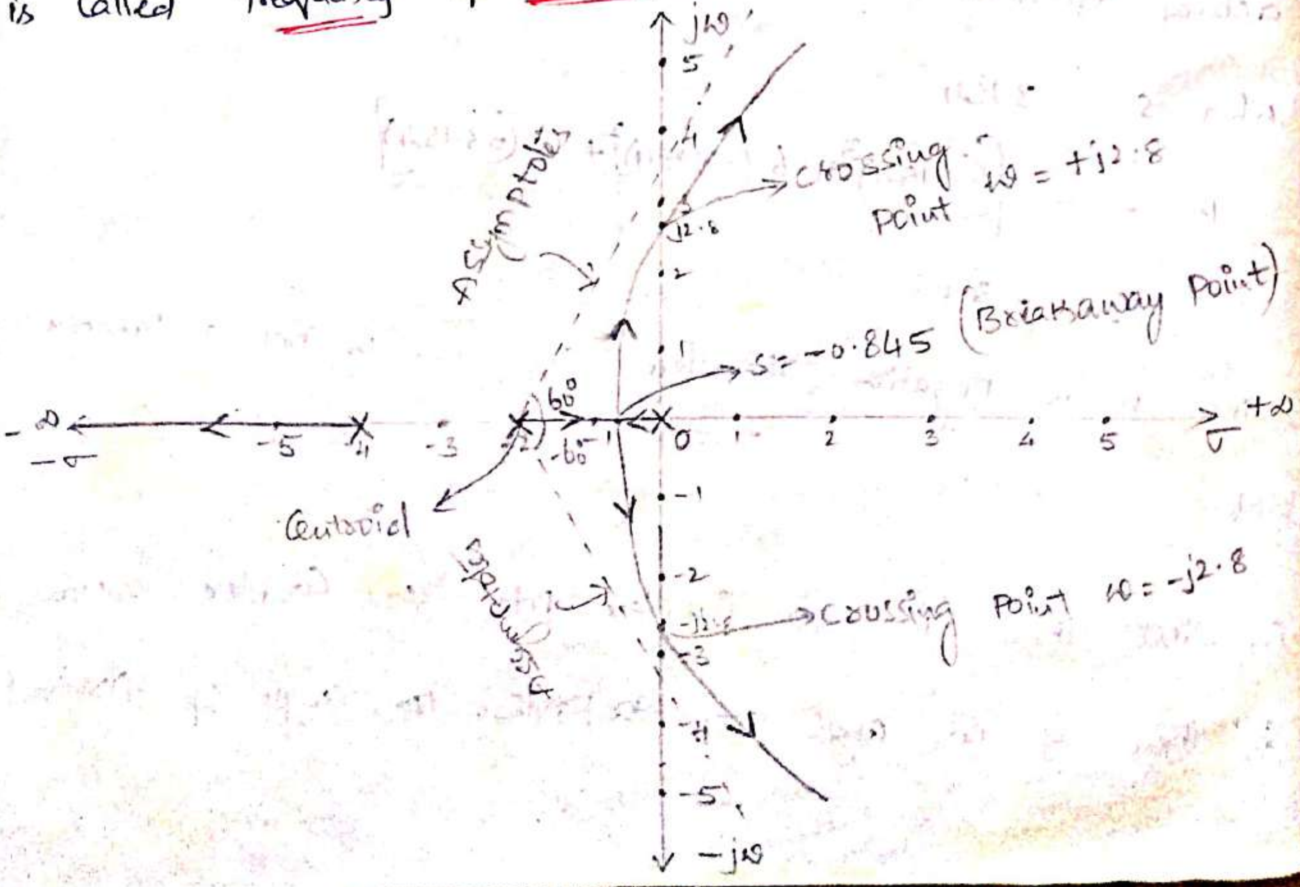
$$6\omega^2=K$$

$$K=6\omega^2$$

$$=6\times 8$$

$$=48$$

The value of $\omega=\pm j2.8$ is the point that crosses the locus on the imaginary axis for $K=48$ and value $\omega=\pm 2.8$ is called frequency of oscillation.



2. Sketch the root locus of the system with loop transfer function $G(s)\,H(s) = \dfrac{K}{s(s+2)(s^2+s+1)}$ $T.F = \dfrac{K(s+9)}{s(s^2+4s+11)}$

Sol given T.F $\Rightarrow G(s)\,H(s) = \dfrac{K}{s(s+2)(s^2+s+1)}$

$$\left[\because \frac{-1 \pm \sqrt{1-4}}{2} \Rightarrow -0.5 \pm j\frac{\sqrt{3}}{2}\right]$$

1. The number of poles are $0, -2, -0.5 \pm j\frac{\sqrt{3}}{2}$

The number of zeros $= 0.$

2. As we got 4 poles so, there must exists 4 Root locus branches.

3. The Angle of Asymptotes $= \dfrac{180^\circ(2q+1)}{n-m}$ $\left[\because q = 0 \ldots n-m = 0,1,2,3,4\right]$

for $q=0 \Rightarrow \dfrac{\pm 180^\circ}{4} = \pm 45^\circ$

$q=1 \Rightarrow \dfrac{180^\circ \times 3}{4-0} = \pm 135^\circ$

$q=2 \Rightarrow \dfrac{5 \times 180^\circ}{4} = \pm 225^\circ.$

$q=3 \Rightarrow \dfrac{7 \times 180^\circ}{4} = \pm 315^\circ.$

4. Centroid $= \dfrac{\text{Sum of the poles} - \text{Sum of zeros}}{\text{no of poles} - \text{no of zeros}}$

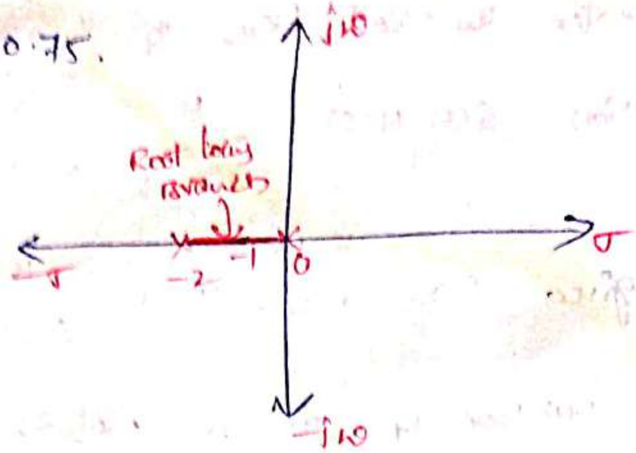
$$= \frac{0 - 2 - 0.5 + j\frac{\sqrt{3}}{2} - 0.5 - j\frac{\sqrt{3}}{2} - 0}{4}$$

Now, taking the only real parts of poles & zeros.

$$\Rightarrow \sigma = \frac{0 - 2 - 0.5 - 0.5 - 0}{4}$$

$$= \frac{-3}{4} = -0.75$$

∴ Centroid $\sigma = -0.75$.



5. Breakaway point

C.E $= 1 + G(s)H(s) = 0$

$$\Rightarrow 1 + \frac{K}{s(s+2)(s^2+s+1)} = 0$$

$$\Rightarrow s(s+2)(s^2+s+1) + K = 0$$

$$\Rightarrow K = -(s^4+3s^3+3s^2+2s) \quad \text{—} \; (A)$$

$$\therefore \frac{dK}{ds} = 0$$

$$\Rightarrow -4s^3-9s^2-6s-2 = 0 \;\Rightarrow\; -(4s^3+9s^2+6s+2) = 0$$

$$\Rightarrow \therefore s = -1.455,\ \cancel{-0.397},\ -0.397+j0.43,\ -0.397-j0.430$$

Substitute $s = -1.455$ in (A)

$$\Rightarrow K = -\left[(-1.455)^4+3(-1.455)^3+3(-1.455)^2+2(-1.455)\right]$$

$$= -\left[4.481 - 9.24 + 6.35 - 2.91\right]$$

$$= -(-1.319) = 1.319.$$

∴ As the 'K' value is positive the root $s = -1.455$ can be a breakaway point

Now, check for $s = -0.397 + j0.430$

$$\cancel{\Rightarrow K = -\left[(-0.397)^4+3(-0.397)^3+3(-0.397)^2+2(-0.397)\right]}$$

$$\cancel{= -\left[0.024 + (-0.187) + 0.472 - 0.794\right]}$$

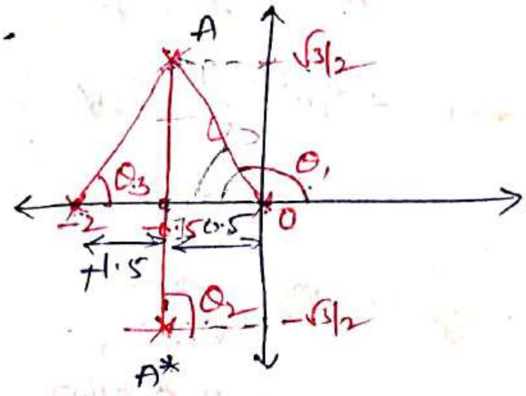
As the Root $s = -0.397 + j0.430$ will not be a breakaway point because this root does not lies in between two poles.

$\therefore$ $s = -1.455$ will be actual Breakaway Point.

6. The angle of Departure from the complex pole

$\Rightarrow$ $s = -0.5 \pm j\frac{\sqrt{3}}{2}$

Now, Draw the Vectors from all the other poles and zeros to the complex pole $s = -0.5 \pm j\sqrt{3}/2$.



$\therefore$ $\theta_1 = 180° - \tan^{-1}\frac{\sqrt{3}}{2\times 0.5}$

$= 120°$.

$\theta_2 = 90°$.

$\theta_3 = \tan^{-1}\frac{\sqrt{3}}{2\times 1.5}$

$= 30°$.

$\therefore$ angle of Departure $\phi_d = 180° - (\theta_1+\theta_2+\theta_3)$

$= 180° - (120+90+30)$

$= 180° - 240°$

$= \underline{-60°}$

Now, the angle of departure from complex pole $s = -0.5 + j\frac{\sqrt{3}}{2}$ is $-60°$. and from $s = -0.5 - j\frac{\sqrt{3}}{2}$ is $-(-60°) = 60°$.

7 Calculating the crossing point.

Characteristic equation is, $1 + G(S)H(S) = 0$

$\Rightarrow$ $1 + \frac{K}{s(s+2)(s^2+s+1)} = 0$

$\Rightarrow$ $s^4 + 3s^3 + 3s^2 + 2s + K = 0.$

Now, Substitute the $s = j\omega$ in above equation we get,

$$\Rightarrow (j\omega)^4 + 3(j\omega)^3 + 3(j\omega)^2 + 2j\omega + K = 0$$

$$\Rightarrow \omega^4 - 3j\omega^3 - 3\omega^2 + 2j\omega + K = 0.$$

Equating the Real & imaginary parts to zero. we get

$$-3j\omega^3 + 2j\omega = 0$$

$$\Rightarrow +3j\omega^{3\,2} = +2j\cancel{\omega}$$

$$\Rightarrow \omega^2 = \frac{2}{3}$$

$$\Rightarrow \omega = \sqrt{\frac{2}{3}}$$

$$= \pm j\,0.8165$$

$j\omega =$

$$\omega^4 - 3\omega^2 + K = 0$$

$$\left(\frac{2}{3}\right)^2 - \cancel{3} \times \frac{2}{\cancel{3}} + K = 0$$

$$\Rightarrow \frac{4 - 9\times 2}{9} + K = 0$$

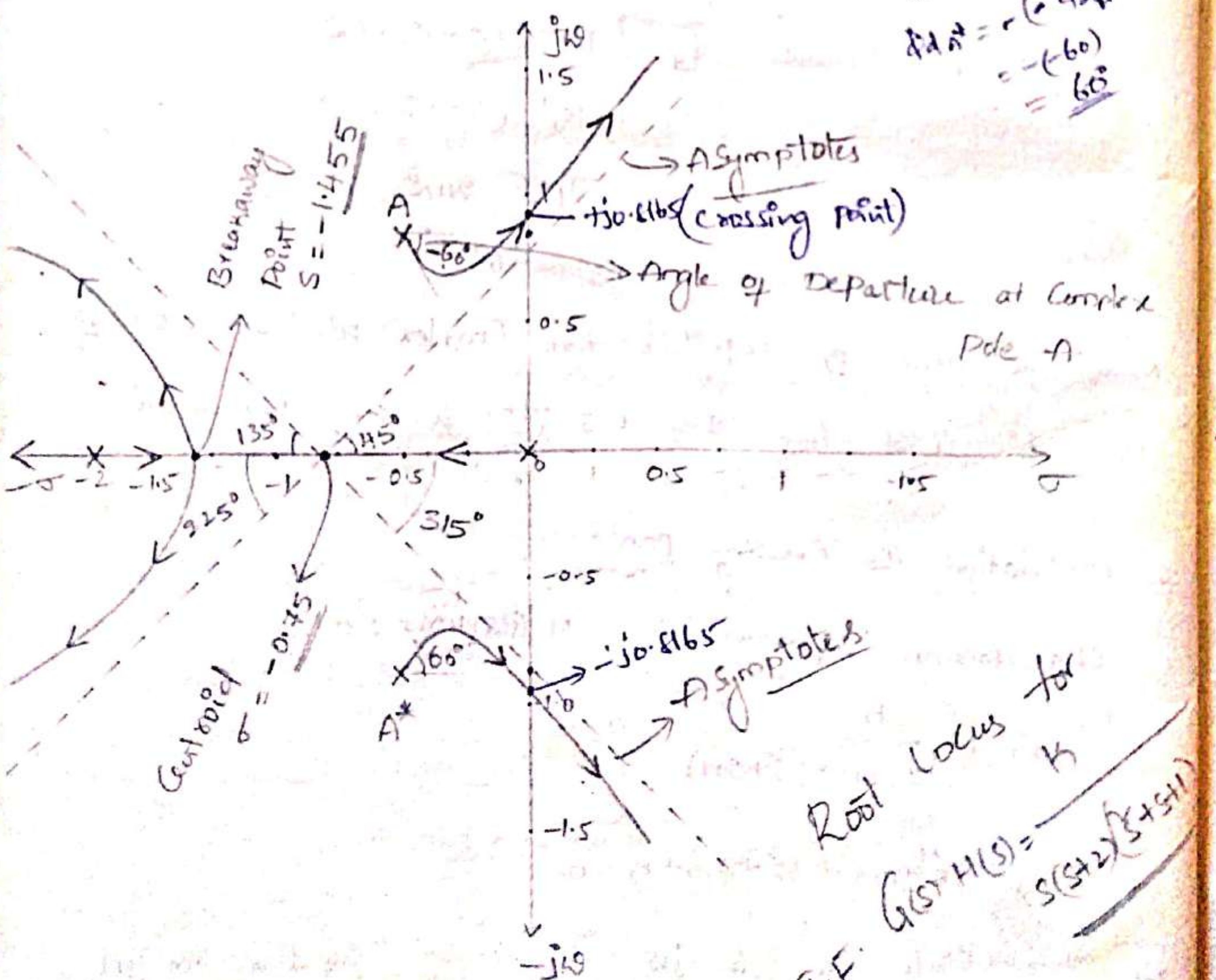
$$\Rightarrow K = \frac{-4+18}{9} = \frac{+14}{9}$$

$$\therefore K = 14/9.$$

Now, plotting the Root locus.

$\phi_A = -60$

$\phi_{A^*} = -(\phi_A)$
$= -(-60)$
$= 60^\circ$



3. Sketch the Root locus for $G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$.

Sol. 1. given, $G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$.

The number of poles $= 0, -1, -2, -3 = 4$

number of zeros $= 0$

2. As we got 4 poles so, there exists 4 Root locus branches.

3. Angle of Asymptotes.

$\Rightarrow \frac{\pm 180(2q+1)}{n-m}$ $\quad q = 0, 1, 2, 3, 4$

for $q=0 \Rightarrow \frac{\pm 180^\circ \times 1}{4} = \pm 45^\circ$.

$q=1 \Rightarrow \frac{\pm 180^\circ \times 3}{4} = \pm 135^\circ$.

$q=2 \Rightarrow \frac{\pm 180^\circ \times 5}{4} = \pm 225^\circ$

$q=3 \Rightarrow \frac{\pm 180^\circ \times 7}{4} = \pm 315^\circ$.

$q=4 \Rightarrow \frac{\pm 180 \times 9}{4} = \mp 45^\circ$.

4. Centroid

$$\sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{number of poles} - \text{number of zeros}}$$

$$= \frac{0-2-1-3-0}{4} = \frac{-6}{4} = \frac{-3}{2}$$

$$= \underline{\underline{-1.5}}.$$

5. finding the Break away point

The C.E $= 1+G(s) = 0 \Rightarrow 1 + \frac{K}{s(s+1)(s+2)(s+3)} = 0$

$\Rightarrow s(s+1)(s+2)(s+3) + K = 0.$

$$\Rightarrow K = -s^4 - 6s^3 - 11s^2 - 6s \quad \text{—— (A)}$$

Now, $$\frac{dK}{ds} = -\left[4s^3 + 18s^2 + 22s + 6\right]$$

and equate $\frac{dK}{ds} = 0$

$$\Rightarrow 4s^3 + 18s^2 + 22s + 6 = 0$$

$$\Rightarrow s = -0.381, \; -2.61, \; -1.5$$

Now Substitute the value $s = -0.381$ in (A)

$$\Rightarrow K = -\left[(-0.381)^4 + 6(-0.381)^3 + 11(-0.381)^2 + 6(-0.381)\right]$$

$$= -\left[(+0.021) - 0.331 + 1.596 - 2.286\right]$$

$$= -(-1) = 1. \quad \text{(Positive)}$$

Now, Check for, $s = -2.61$

$$\Rightarrow K = -\left[(-2.61)^4 + 6(-2.61)^3 + 11(-2.61)^2 + 6(-2.61)\right]$$

$$= -\left[46.4 - 106.67 + 74.933 - 15.66\right]$$

$$= -(-0.997) \cong 1. \quad \text{(Positive)}$$

for $s = -1.5$

$$\Rightarrow K = -\left[(-1.5)^4 + 6(-1.5)^3 + 11(-1.5)^2 + 6(-1.5)\right]$$

$$= -\left[5.06 - 20.25 + 24.75 - 9\right]$$

$$= -[0.56] = -0.56. \quad \text{(Negative)}$$

Now, for the Roots $s = -0.381$ and $s = -2.61$ the 'K' value is positive and for $s = -1.5$ it is Negative

Therefore, $s = -0.381$ and $s = -2.61$ are the actual Breakaway points.

6. Crossing point.

$$\Rightarrow s^4 + 6s^3 + 11s^2 + 6s + K = 0.$$

Now, Substitute $s = j\omega$ in above equation,

$$\Rightarrow (j\omega)^4 + 6(j\omega)^3 + 11(j\omega)^2 + 6j\omega + K = 0.$$

$$\Rightarrow \omega^4 - 6j\omega^3 - 11\omega^2 + 6j\omega + K = 0.$$

$$\Rightarrow \omega^4 - 11\omega^2 + K = 0$$

$$1 - 11 + K = 0$$

$$K = 10.$$

$$-6j\omega^3 + 6j\omega = 0.$$

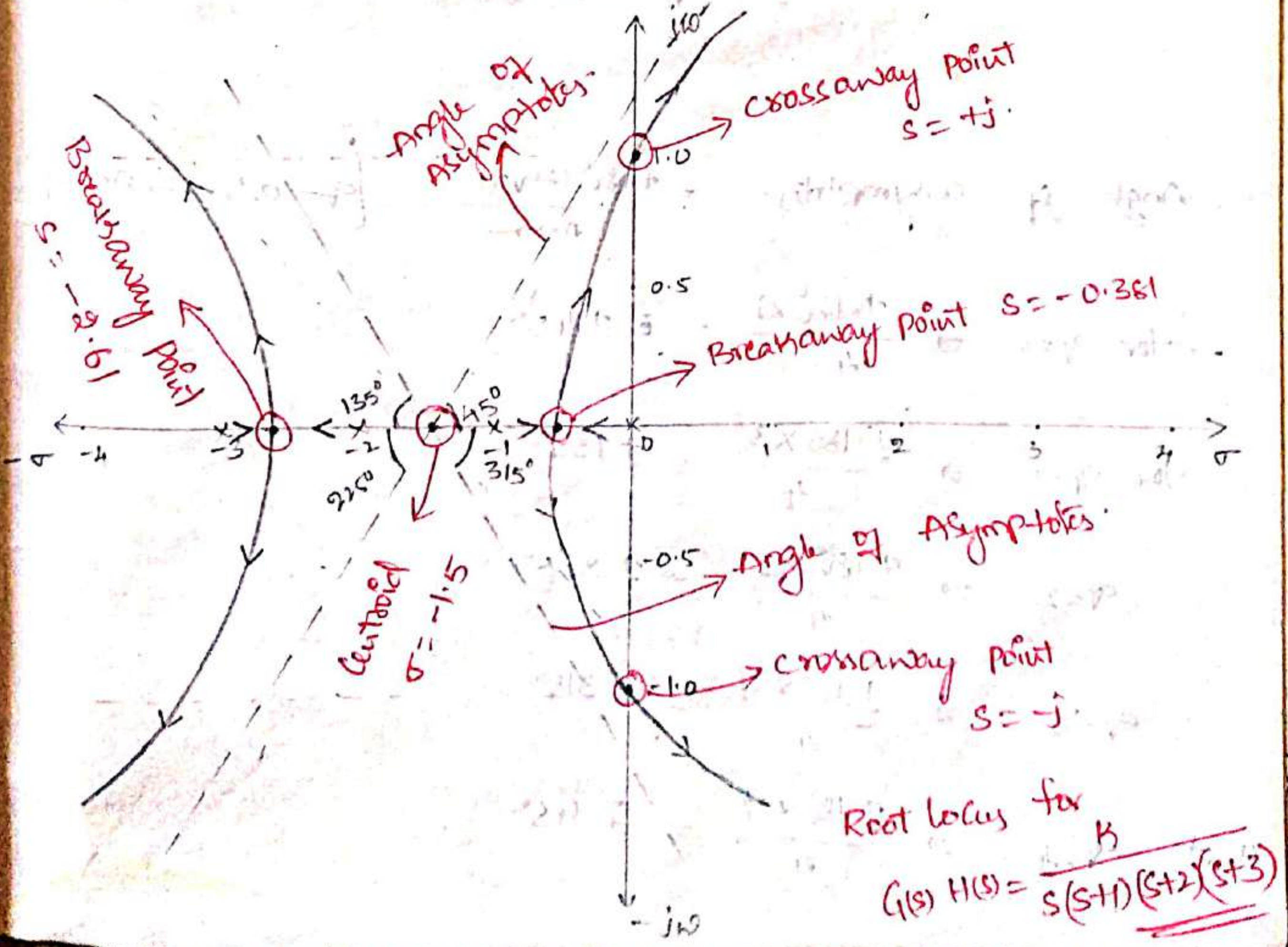
$$\cancel{-6j}\omega^3 = -\cancel{6j}\omega$$

$$\omega^2 = 1$$

$$\omega = \pm\sqrt{1}$$

$$= \pm 1.$$

Now, the Complete Root locus is,



Root locus for $G(s)H(s) = \dfrac{K}{s(s+1)(s+2)(s+3)}$

(4) Sketch the Root locus for unity feed back system

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}.$$

Sol: given, $G(s)H(s) = \dfrac{K}{s(s+4)(s^2+4s+20)}$.

$$\left[\frac{-4 \pm \sqrt{16-80}}{2} \Rightarrow -2 \pm j4\right]$$

1. Number of poles $= 0, -4, -2 \pm j4$.

$= 4$.

Number of zeros $= 0$.

2. As we got the 4 poles so, there should be 4 Root locus Branches, will exist.

3. Centroid $= \sigma = \dfrac{\text{Sum of poles} - \text{Sum of zeros}}{\text{no of poles} - \text{no of zeros}}$

$$= \frac{-4-2+j4-2-j4-0}{4}$$

$$= \frac{-8}{4} = -2$$

$\therefore \sigma = -2$

4. Angle of assymptotes $= \dfrac{\pm 180^\circ(2q+1)}{n-m}$ $\quad [q = 0, 1, \ldots, n-m]$

$\therefore$ for $q=0 \Rightarrow \dfrac{\pm 180^\circ \times 1}{4} = \pm 45^\circ$

for $q=1 \Rightarrow \dfrac{\pm 180^\circ \times 3}{4} = \pm 135^\circ$

$q=2 \Rightarrow \dfrac{\pm 180^\circ \times 5}{4} = \pm 225^\circ$.

$q=3 \Rightarrow \dfrac{\pm 180^\circ \times 7}{4} = \pm 315^\circ$

and, $q=4 \Rightarrow \dfrac{\pm 180^\circ \times 9}{4} = \mp 45^\circ$.

5. Breakaway point

The characteristic equation, $1+G(s)H(s)=0$.

$$\Rightarrow 1+\frac{K}{s(s+4)(s^2+4s+20)}=0$$

$$\Rightarrow s(s+4)(s^2+4s+20)+K=0.$$

$$\Rightarrow K = -\left[(s^2+4s)(s^2+4s+20)\right] \quad \text{--- (1)}$$

$$\Rightarrow K = -(s^4+8s^3+36s^2+80s). \quad \text{--- (A)}$$

Differentiate above equation w.r.t. s. we get,

$$\frac{dK}{ds} = -(4s^3+24s^2+72s+80) = 0$$

$$\Rightarrow 4s^3+24s^2+72s+80=0$$

on dividing by 4 we get

$$\Rightarrow s^3+6s^2+18s+20=0.$$

The Roots for above equation can also be find by trail and Error Method (or) from Calculator.

$$s = -2, \quad -2 \pm j2.449.$$

Here, we got the roots as Complex except $s=-2$, and now checking for

∴ ~~so~~ ~~neglecting~~ ~~the~~ ~~Complex~~ ~~roots~~ ~~and~~

even. for $-2 \pm j2.449$ K = positive

the root $s=-2$ in

$$\Rightarrow K = -\left[s^4+8s^3+36s^2+80s\right]$$

$$= -\left[(-2)^4+8(-2)^3+36\times4+80\times-2\right]$$

$$= -\left[16-64+144-160\right]$$

$$= -\left[-64\right] = 64. \text{ (positive)}$$

When $s = -2$, $-2 \pm j2.45$ we got the value of K as positive so, the actual breakaway point is $s = -2$, $-2 \pm j2.45$.

6. As we got a complex pole so, there is a need to find (a) to calculate angle of departure.

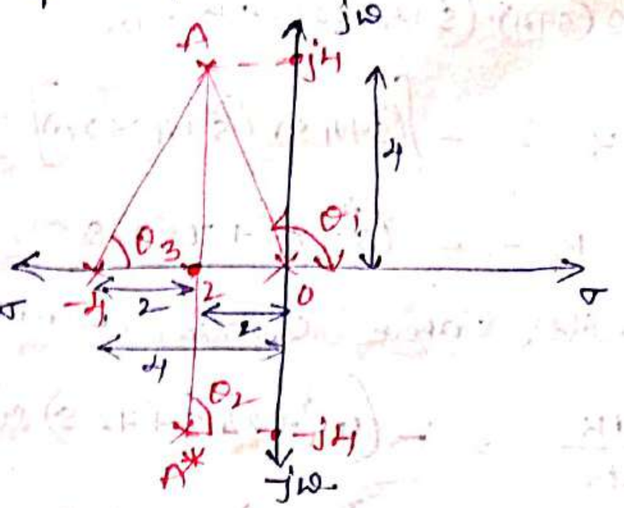
$\theta_1 = 180° - \tan^{-1}\left(\frac{4}{2}\right)$

$\therefore \theta_1 = 117°$

$\theta_2 = 90°$

$\theta_3 = \tan^{-1}\left(\frac{4}{2}\right)$

$\therefore \theta_3 = 63°$



Now, the angle of departure ϕ_d is given,

$$\phi_d = 180° - [\theta_1 + \theta_2 + \theta_3]$$
$$= 180° - [117 + 90 + 63]$$
$$= 180° - 270°$$
$$\Rightarrow \phi_d = -90°$$

$\therefore$ The angle of departure from the complex pole is $-90°$.

for the complex pole $A^* = -(-90°)$
$= 90°$.

7. Crossing point on imaginary axis.

from the characteristic equation:

$$\Rightarrow s^4 + 8s^3 + 36s^2 + 80s + K = 0.$$

Now, Substitute $s = j\omega$ in above equation,

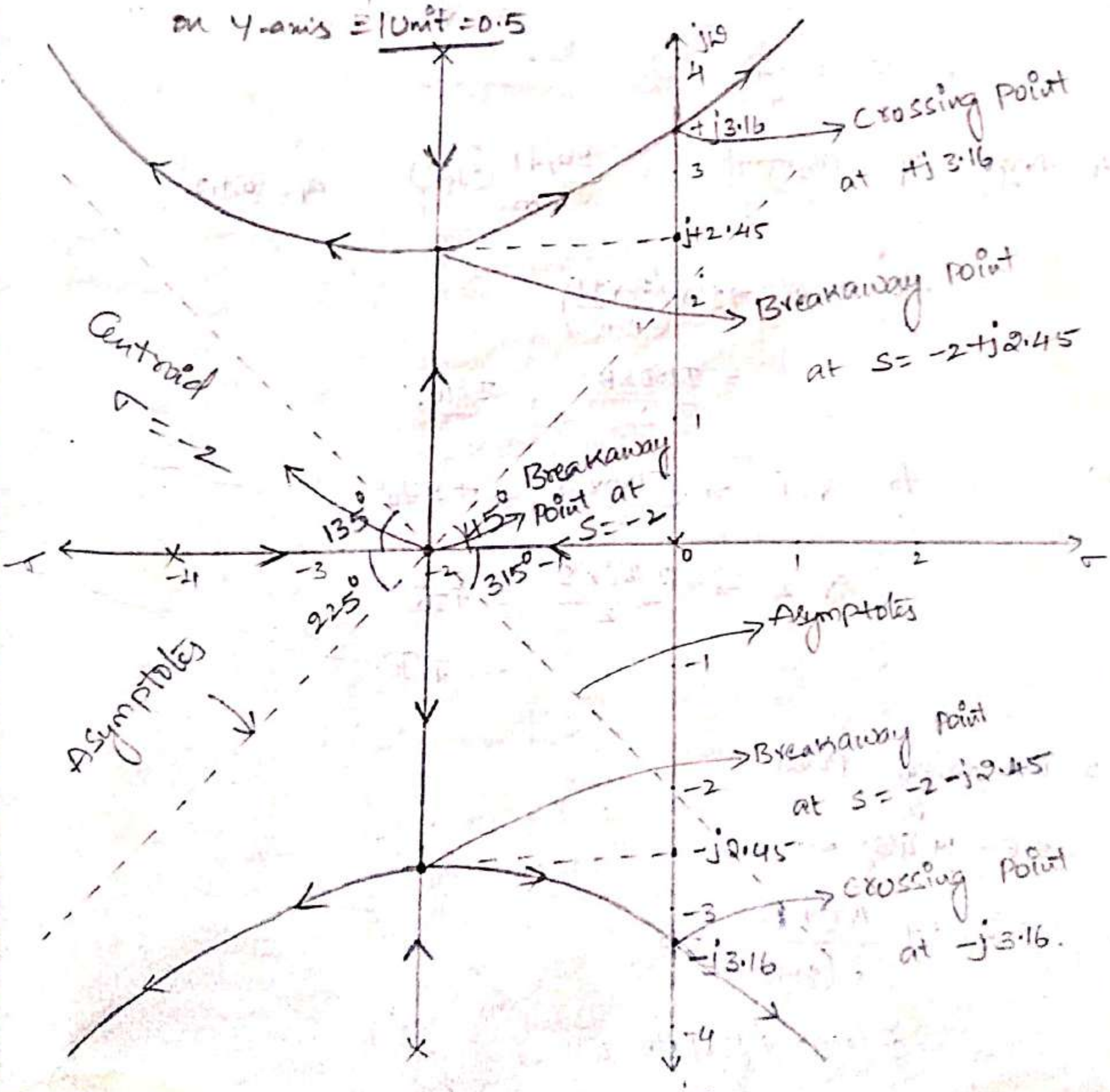
$$\Rightarrow (j\omega)^4 + 8(j\omega)^3 + 36(j\omega)^2 + 80j\omega + K = 0.$$

$$\Rightarrow \omega^4 - 8j\omega^3 - 36\omega^2 + 80j\omega + K = 0.$$

Now, Equate the real part and imaginary part to zero.

$\Rightarrow \omega^4 - 36\omega^2 + K = 0$	$-8j\omega^3 + 80j\omega = 0$
$(10)^2 - 36\times 10 + K = 0$	$\Rightarrow -\cancel{8j}\omega^{\cancel{3}\,2} = -\frac{\cancel{80j\omega}}{10}$
$100 - 360 + K = 0$	$\Rightarrow \omega^2 = 10.$
$\Rightarrow K = \underline{260}.$	$\Rightarrow \omega = \pm\sqrt{10}$
	$\boxed{\omega = \pm 3.16}$

Scale: on x-axis &
on y-axis = $\underline{1 \text{ unit} = 0.5}$



(5) $G(s) = \frac{K(s+1)}{s^2(s+9)}$.

Sol: given, $G(s) = \frac{K(s+1)}{s^2(s+9)}$

1. The number of Poles $= 0, 0, -9$.

$= \underline{3}$.

Number of zeros $= \underline{-1}$.

2. As we got 3 Poles So, the 3 Root locus branches will exist

3. Centroid $= \sigma = \frac{\text{Sum of poles} - \text{Sum of zeros}}{\text{No of poles} - \text{no of zeros}}$

$= \frac{0-0-9+1}{3-1} = \frac{-8}{2}$

$= -4$.

$\therefore \underline{\underline{\sigma = -4}}$.

4. Angle of asymptotes $= \frac{2q+1}{n-m}(\pm 180^\circ)$ $q = 0, 1, 2$

for $q=0 \rightarrow \pm 180\left(\frac{2q+1}{n-m}\right)$

$= \frac{\pm 180 \times 1}{2} = \underline{\pm 90^\circ}$.

for $q=1 \Rightarrow \frac{\pm 180 \times 3}{2} = \underline{\underline{\pm 270^\circ}}$

$q=2 \Rightarrow \frac{\pm 180^\circ \times 5}{2} = 450^\circ$

$= \underline{\mp 90^\circ}$.

5. Breakaway Point

$C.E = 1 + G(s) = 0$

$= 1 + \frac{K(s+1)}{s^2(s+9)} = 0$

$= s^2(s+9) + Ks + K = 0$

$$\Rightarrow K = -\left[\frac{s^2(s+9)}{s+1}\right] \quad \text{— (A)}$$

$\therefore$ Now, $\frac{dK}{ds} = 0$

$$\Rightarrow \frac{dK}{ds} = -\left[\frac{s^2(s+9) - (s+1)(3s^2+18s)}{(s+1)^2}\right] = 0$$

$$= -\left[\frac{s^3+9s^2-(3s^3+18s^2+3s^2+18s)}{(s+1)^2}\right] = 0$$

$$= -\left[\frac{-2s^3-12s^2-18s}{(s+1)^2}\right] = 0$$

$$\Rightarrow 2s^3+12s^2+18s = 0.$$

$$s = 0, 0, -3.$$

Now, Substitute $s = -3$ in (A)

$$\Rightarrow K = -\left[\frac{s^2(s+9)}{s+1}\right]$$

$$= -\left[\frac{9(-3+9)}{-3+1}\right]$$

$$= +\frac{9(6)^3}{\cancel{2}}$$

$$K = 27 \quad \text{(positive).}$$

$\therefore$ when $s = -3$, the K value is positive. So, the root $s = -3$ is the actual breakaway point

6. As there is no Complex pole. Hence, there is no need to find the angle of departure (or) angle of Arrival.

7. Crossing point

from the C.E $\Rightarrow$ $1+G(s)H(s)=0$

$$\Rightarrow 1+\frac{K(s+1)}{s^2(s+9)}=0$$

$$\Rightarrow s^2(s+9)+Ks+K=0$$

$$\Rightarrow s^3+9s^2+Ks+K=0$$

Now, Substitute $s=j\omega$ in above equation,

$$\Rightarrow (+j\omega)^3+9(j\omega)^2+Kj\omega+K=0$$

$$\Rightarrow -j\omega^3-9\omega^2+Kj\omega+K=0$$

$$\Rightarrow -j\omega^3+Kj\omega=0 \qquad\qquad -9\omega^2+K=0$$

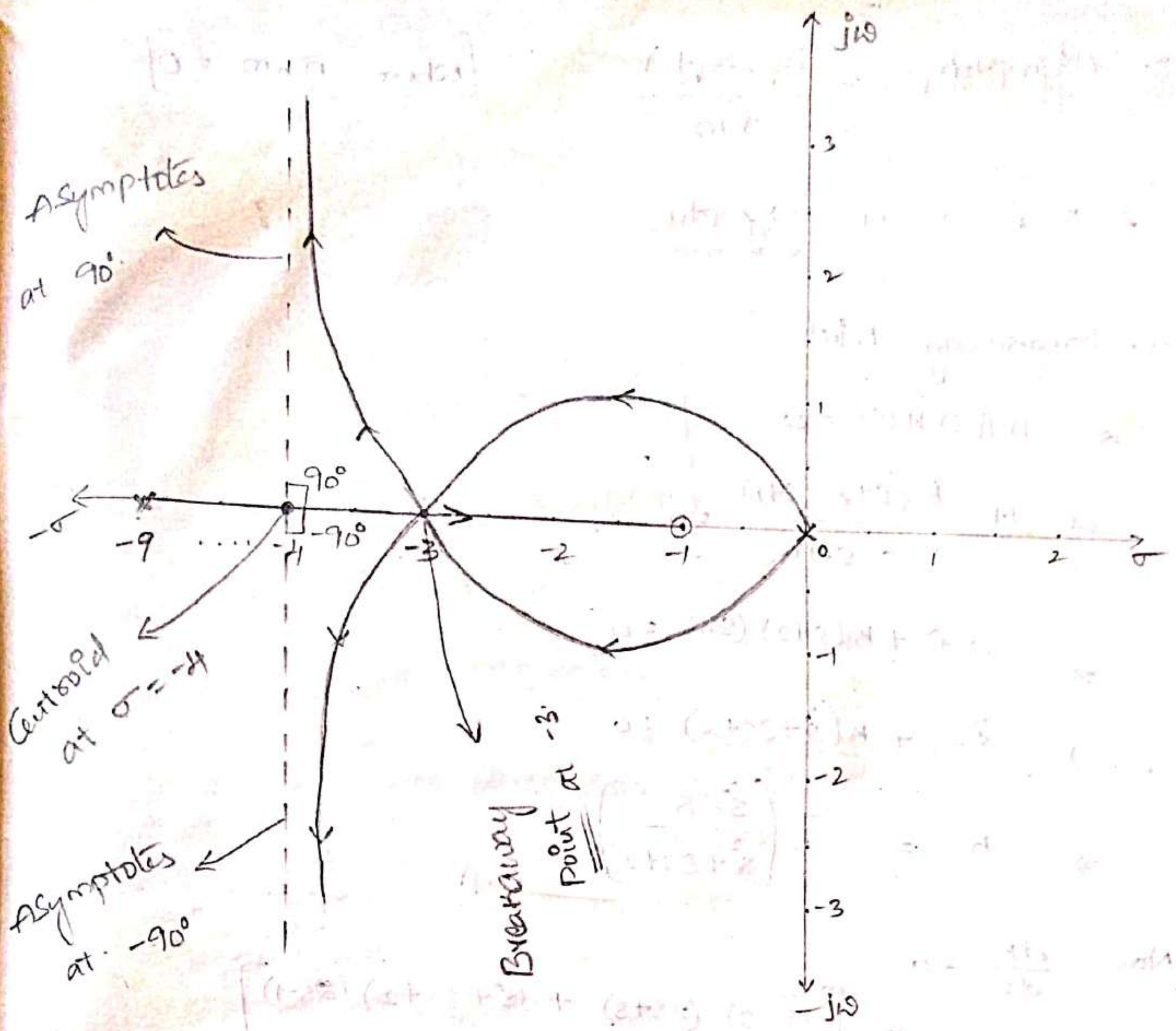
$$\Rightarrow \cancel{-j}\omega^{3\,2}=-K\cancel{j\omega} \qquad\qquad \Rightarrow -9(K)+K=0$$

$$= \omega^2=K. \qquad\qquad \Rightarrow K=9K$$

$\Rightarrow$

$\therefore$ where 'K' is the gain of the System and this gain of the System Can't be zero.

Hence, there is no intersection point on imaginary axis.



(5) Sketch the Root locus for a Transfer function.

$$G(s) = \frac{K(s+2)(s+1)}{s(s-1)}.$$

Sol: given, $G(s) = \frac{K(s+2)(s+1)}{s(s-1)},$

1. Number of poles = 0, 1.

   Number of zeros = -2, -1.

2. As there are two poles so, there exists two Root locus branches will exist on the Real axis.

3. Asymptotes $= \frac{180^\circ(2q+1)}{n-m}$ [where $n-m=0$]

$\therefore$ There is no Asymptotes

4. Breakaway point

$\Rightarrow \quad 1 + G(s)H(s) = 0$

$\Rightarrow \quad 1 + \frac{K(s+2)(s+1)}{s(s-1)} = 0$

$\Rightarrow \quad s(s-1) + K(s+2)(s+1) = 0$

$\Rightarrow \quad s^2 - s + K(s^2+3s+2) = 0$

$\Rightarrow \quad K = -\left[\frac{s^2-s}{s^2+3s+2}\right]$ — (A)

Now, $\frac{dK}{ds} = 0$

$\Rightarrow \frac{dK}{ds} = -\left[\frac{-(s^2-s)(2s+3) + (s^2+3s+2)(2s-1)}{(s^2+3s+2)^2}\right]$

$= -\left[\frac{-2s^3 - s^2 + 3s + 2s^3 + 5s^2 + s - 2}{(s^2+3s+2)^2}\right] = 0$

$\left[\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}\right]$

$\Rightarrow \quad -\left[\frac{2(2s^2+2s-1)}{(s^2+3s+2)^2}\right] = 0$

$\Rightarrow \quad 2(2s^2+2s-1) = 0$

$\Rightarrow \quad 2s^2+2s-1 = 0$

$\Rightarrow \quad s = 0.36, -1.36$

Substitute $s = -1.36$ in (A)

$\Rightarrow \quad K = \left[\frac{-(s^2-s)}{s^2+3s+2}\right]$

$$K = -\left[\frac{(-1.36)^2 + 1.36}{(-1.36)^2 + 3(-1.36) + 2}\right]$$

$$\Rightarrow K = -\left[\frac{3.206}{-0.2304}\right]$$

$$= -(-13.93)$$

$$\therefore K = 13.93 \quad \text{(positive)}$$

Now, for $s = 0.36$

$$K = -\left[\frac{(0.36)^2 - 0.36}{(0.36)^2 + 3(0.36) + 2}\right]$$

$$= -\left[\frac{0.1296 - 0.36}{0.1296 + 1.08 + 2}\right]$$

$$= -\left[\frac{-0.2304}{3.2096}\right]$$

$$\therefore K = 0.071 \quad \text{(positive)}$$

$\therefore$ $S = 0.36$ is lies between the two poles 0 and 1 so it is called as Break away point, where as the $s = -1.36$ lies between two zeros. So, it is called as Break in point

5. As we don't have a complex pole so, no need to find the angle of departure

6. Crossing Point

$\Rightarrow s^2 - s + Ks^2 + 3Ks + 2K = 0$

Now, Substitute $s = j\omega$ in above equation,

$\Rightarrow (j\omega)^2 - j\omega + K(j\omega)^2 + 3Kj\omega + 2K = 0.$

$\Rightarrow -\omega^2 - K\omega^2 + 2K = 0$

$-\omega^2(1+K) + 2K = 0$

$\Rightarrow \omega^2(1+K) = 2K$

$\Rightarrow \omega^2 = \frac{2K}{1+K}$

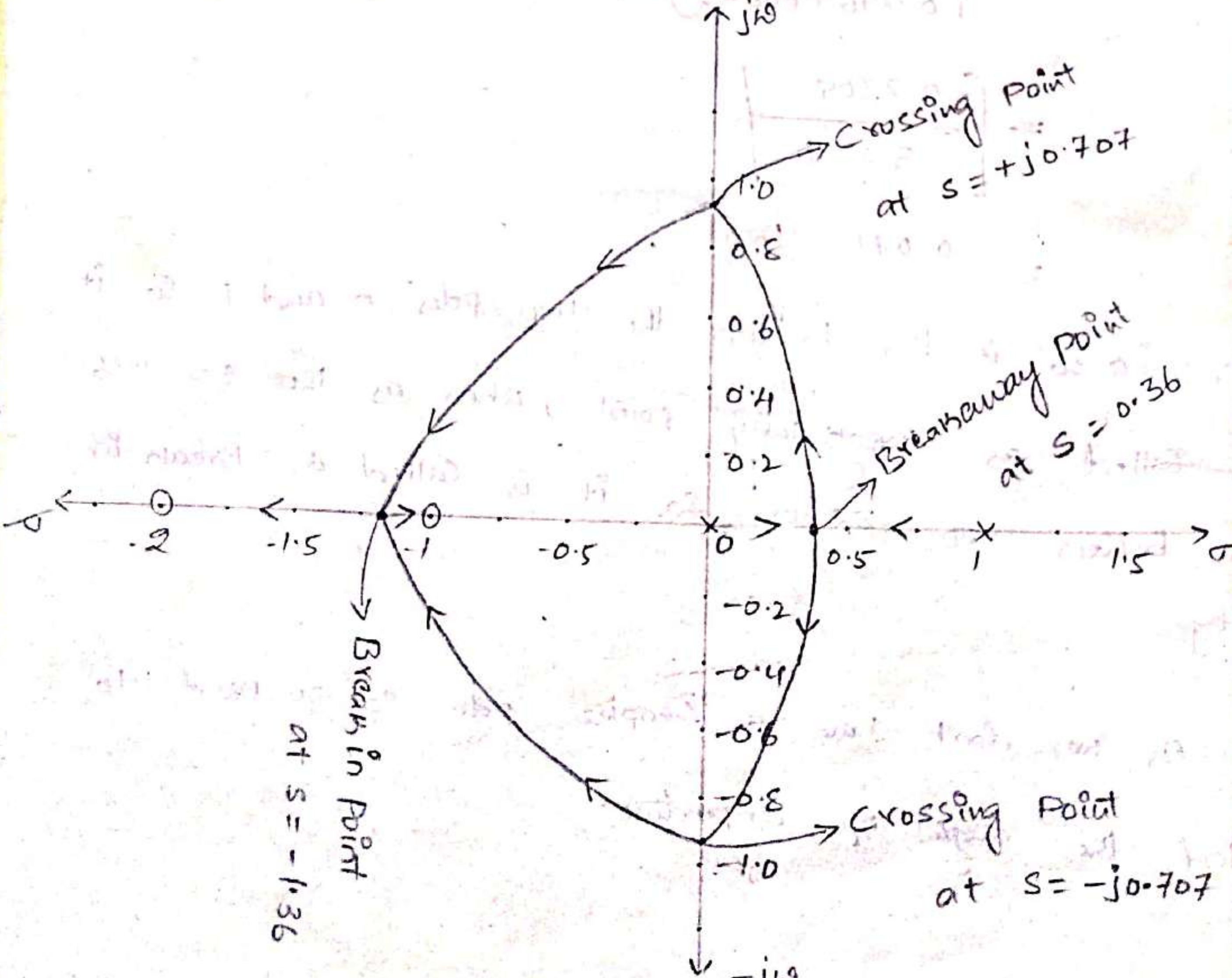
$= \frac{2/3}{1+1/3} = 0.5$

$\Rightarrow \omega = \pm\sqrt{0.5} = \pm 0.7$

$-\omega + 3K\omega = 0$

$\Rightarrow \omega = 3K\omega$

$\Rightarrow K = 1/3.$



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UNIT-5

BODE PLOTS.

* Frequency Response ;

The frequency Response is the steady state response of a system when the input to the system is a sinusoidal signal.

Consider a linear time invariant (LTI) system, H. let x(t) be an input sinusoidal signal. The Response or output y(t) is also a sinusoidal signal of same frequency but with different magnitude and phase angle.

The frequency Response of a system is normally obtained by varying the frequency of the input signal by keeping the magnitude of the input signal at a constant value.

In the system transfer function T(S), if 's' is Replaced by $j\omega$ then the Resulting transfer function $T(j\omega)$ is called Sinusoidal Transfer function.

* Advantages of FRequency Response Analysis.

1. The frequency Response analysis and designs can be extended to certain non-linear control systems.
2. The Transfer function of the complicated systems can be determined experimentally by frequency response analysis.
3. The design and parameter adjustment of the open loop transfer function of a system for specified closed loop performance is carried out more easily in the frequency domain.

* frequency Domain Specifications :

The Performance and characteristics of a System in frequency domain are measured in terms of frequency domain specifications.

The frequency domain specifications are,

1. Resonant Peak, M_r
2. Resonant Frequency, ω_r
3. Band width
4. cut-off rate
5. gain Margin
6. phase Margin.

1. The Maximum Value of the magnitude of the closed loop transfer function is called Resonant peak, M_r. $\boxed{M_r = \frac{1}{2\zeta\sqrt{1-\zeta^2}}}$

2. The frequency at which the Resonant peak occurs is called Resonant frequency ω_r. $\boxed{\omega_r = \omega_n\sqrt{1-2\zeta^2}}$.

3. The Bandwidth is the range of frequencies for which the System gain is more than $-3dB$. The frequency at which the gain is $-3dB$ is called cut-off frequency.

4. The Slope of the log-magnitude curve near the cut-off frequency is called the cut-off Rate. The cut-off rate Indicates the ability of the system to differentiate the signal from the noise

5. The gain margin k_g, is defined as the Reciprocal of the magnitude of the open loop transfer function at phase cross over frequency. (ω_{pc}).

The frequency at which the phase of open loop transfer function is 180° is called phase cross over frequency ω_{pc}

Gain Margin $k_g = \frac{1}{|G(j\omega_{pc})|}$.

The gain margin in dB is expressed as,

$$k_g = 20 \log k_g \text{ dB}$$
$$= 20\log \frac{1}{|G(j\omega_{pc})|} = -20\log |G(j\omega_{pc})|.$$

5. The phase margin γ, is that amount of additional phase lag at gain cross over frequency required to bring the system to the verge of instability.

The gain cross over frequency is the frequency at which the magnitude of the open loop transfer function is ~~unity~~ (or) zero dB is unity.

The phase margin, is obtained by adding $180°$ to the phase angle ~~φ~~ of the open loop transfer function at the gain cross over frequency.

phase margin $\boxed{\gamma = 180° + \phi_{gc}}$

$$\Rightarrow \boxed{\gamma = 180° + \phi_{gc}}$$

* Bode Plots :-

The Bode plot is a frequency response plot of the transfer function of a system. A Bode plot consists of two graphs. One is the magnitude plot and the other is the phase plot.

The Bode plot can be drawn for both open loop and closed loop systems, usually, the Bode plot is drawn for open loop systems.

The standard Representation of the logarithmic magnitude of the open loop transfer function $G(j\omega)$ is $20 \log |G(j\omega)|$.

The main advantage of Bode plot is multiplication of magnitudes can be converted to Addition.

* Step-By-Step Procedure for plotting the Magnitude Plot:

Step-1:- Convert the transfer function into Bode form or time Constant form. The Bode form of the transfer function is,

$$G(s) = \frac{K(1+sT_1)}{s(1+sT_2)(1+sT_3)}$$

$$G(j\omega) = \frac{(1+j\omega T_1)K}{j\omega(1+j\omega T_2)(1+j\omega T_3)}.$$

Step-2 :- List the Corner frequency in the increasing order and Prepare a table as shown below.

Term	Corner frequency	Slope	Change in slope

For the frequency factor $(1+j\omega T)$, the frequency $\left(\omega = \frac{1}{T}\right)$ is called Corner frequency.

In the above table enter K (or) $K/(j\omega)^n$ (or) $K(j\omega)^n$ as the first term and the other terms in the increasing order of corner frequencies. Then, enter the corner frequency, slope contributed by each term and change in slope at every corner frequency.

Step-3:- Choose an arbitrary frequency ω_ℓ which is lesser than the lowest corner frequency. Now, calculate the dB magnitude of K (or) $K/(j\omega)^n$, (or) $K(j\omega)^n$ at ω_ℓ and at the lower corner frequency.

Step. 4 :- Then calculate the gain at every corner frequency one by one by using the formula.

$$\text{Gain at } \omega_y = \text{change in gain from } \omega_x \text{ to } \omega_y + \text{Gain at } \omega_x$$

$$= \left[\text{Slope from } \omega_x \text{ to } \omega_y \times \log \frac{\omega_y}{\omega_x}\right] + \text{Gain at } \omega_x$$

Step 5 :- Choose an arbitrary frequency ω_h which is greater than the highest corner frequency. Calculate the gain at ω_h by using the formula at in step -4.

Step. 6 :- Now, in a Semilog graph, take the Required range of frequency on X-axis and the range of dB Magnitude on Y-axis. Then Mark all points obtained from steps 3, 4, 5 on the graph and join the points by straight lines.

* Procedure for the phase plot of Bode plot :-

Take another axis in the graph on Y-axis where the magnitude plot is drawn and in this Y-axis mark the desired range of phase angles after choosing proper scale. From the tabulated values of ω and phase angles, mark all the points on the graph. Join the points by a smooth curve.

* Problems:-

1. Sketch the Bode plot for the following transfer function

$$G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}.$$

Sol: given transfer function $G(s) = \frac{10}{s(1+0.4s)(1+0.1s)}$

The Sinusoidal transfer function is obtained by Replacing $j\omega$ in the place of s.

$$\Rightarrow G(j\omega) = \frac{10}{j\omega(1+0.4j\omega)(1+0.1j\omega)}.$$

Magnitude plot:-

The corner frequencies are $\omega = \frac{1}{T}$

$\Rightarrow \omega_{c_1}$ for $(1+0.4j\omega) = \omega_{c_1} = \frac{1}{0.4} = 2.5$ rad/sec

ω_{c_2} for $(1+0.1j\omega) = \omega_{c_2} = \frac{1}{0.1} = 10$ rad/sec.

Now, the various terms are listed in the table in the increasing order as shown below,

Term	Corner frequency rad/sec	Slope dB/sec	change in slope dB/sec
$\frac{10}{j\omega}$	—	-20	—
$\frac{1}{1+j0.4\omega}$	$\omega_{c_1} = \frac{1}{0.4} = 2.5$	-20	$-20-20 = -40$
$\frac{1}{1+j0.1\omega}$	$\omega_{c_2} = \frac{1}{0.1} = 10$	-20	$-40-20 = -60$

Now, choose a low frequency ω_l such that $\omega_l < \omega_{c_1}$ and choose a high frequency ω_h such that $\omega_h > \omega_{c_2}$.

let, $\omega_l = 0.1$ rad/sec, and, $\omega_h = 50$ rad/sec let $A = |G(j\omega)|$ in db.

let Calculate Magnitudes A at $\omega_l, \omega_{c_1}, \omega_{c_2}, \omega_h$.

At $\omega = \omega_l \Rightarrow A = 20\log\left|\left(\frac{10}{j\omega}\right)\right| = 20\log\frac{10}{\omega} = 20\log\left(\frac{10}{0.1}\right)$

$= 40\,db.$

$\frac{\sqrt{10^2}}{\sqrt{0+\omega^2}}\left(\frac{10}{\omega}\right)$ $20\log\left(\frac{10}{\omega}\right)$

At $\omega = \omega_{c_1} \Rightarrow A = 20\log\left(\frac{10}{\omega}\right) = 20\log\left(\frac{10}{2.5}\right)$

$= 12\,db.$

At $\omega = \omega_{c_2} \Rightarrow A = \left(\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right) + A_{\omega=\omega_{c_1}}$

$= -40 \times \log\left(\frac{10}{2.5}\right) + 12$

$= -12\,db.$

At $\omega = \omega_h \Rightarrow A = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c_2}}\right] + A_{\omega=\omega_h}.$

$= -60 \times \log\left(\frac{50}{10}\right) - 12$

$= -54\,db.$

Phase plot :-

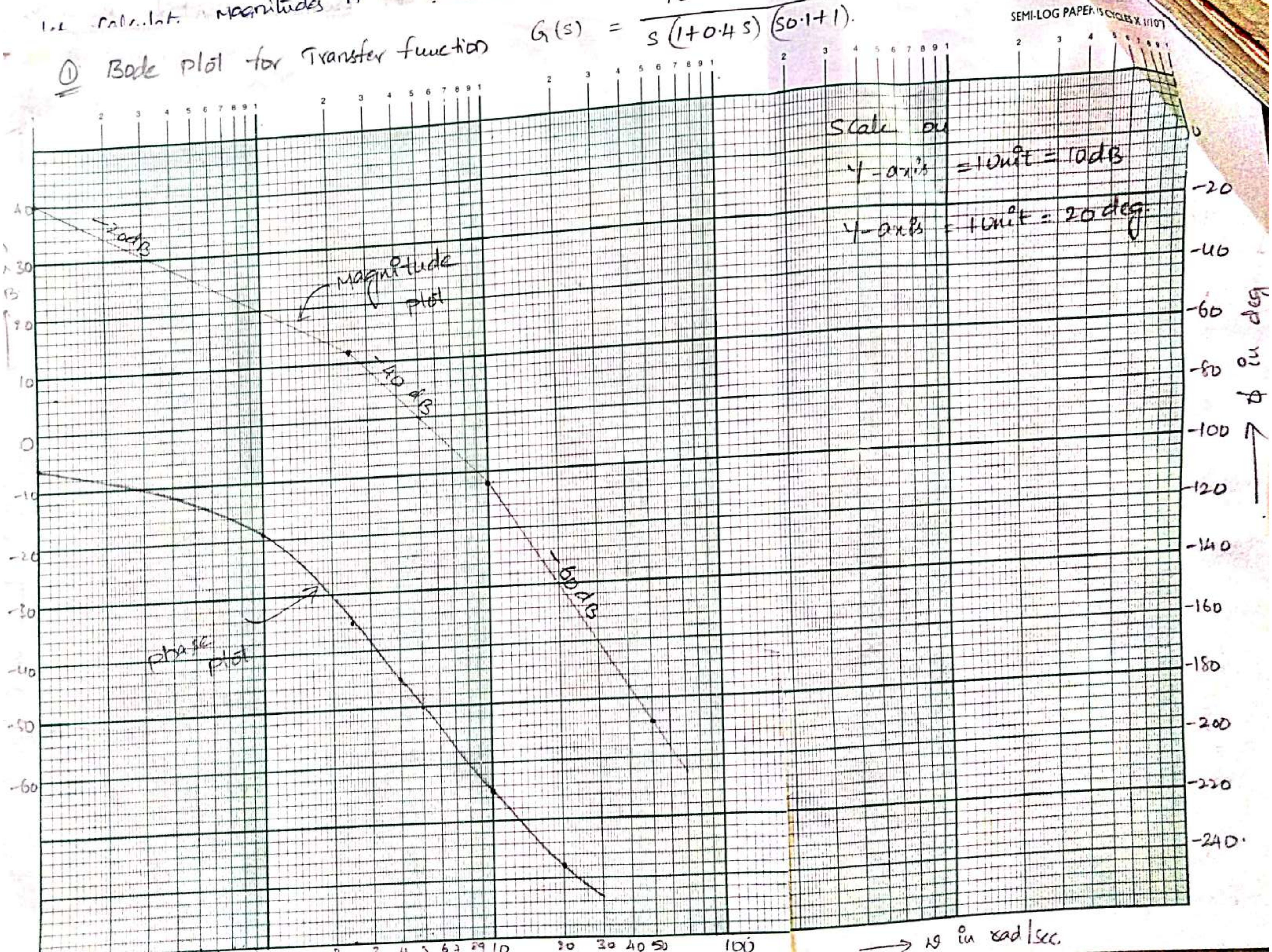
The phase angle of $G(j\omega)$ as a function of ω is given by,

$\phi = -90° - \tan^{-1}0.4\omega - \tan^{-1}0.1\omega$

ω rad/sec	$\tan^{-1}0.4\omega$ in deg	$\tan^{-1}0.1\omega$ in deg.	$\phi = \angle G(j\omega)$ in deg.
0.1	2.29	0.57	$-92.86 \simeq -93$
1	21.80	5.71	$-117.5 \simeq -118$
2.5	45.0	14.0	-149
4	57.99	21.88	$-169.79 \simeq -170$
10	75.96	45.0	$-210.96 \simeq -211$
20	82.87	63.43	$-236.3 \simeq -236.$

Now, Plot the magnitude and the phase plots on the same semi-log graph sheets by taking the proper scale on the graph.

① Bode plot for Transfer function $G(s) = \frac{10}{s(1+0.4s)(s0.1+1)}$.



Gain cross over frequency : The gain crossover frequency ω_{gc} is defined as the frequency at which the Resultant Magnitude is 0dB.,
ie, the gain crossover frequency is defined as the frequency at which $|G(j\omega)\,H(j\omega)| = 1$

phase crossover frequency : The phase crossover frequency ω_{pc} is defined as the frequency at which the Resultant phase is -180°
ie, the phase crossover frequency is defined as the frequency at which $|G(j\omega)\,H(j\omega)| = -180^\circ$.

2. Sketch the Bode Plot for the following transfer function

$$G(s)\,H(s) = \frac{Ks^2}{(1+0.25s)\,(1+0.025s)}$$, find K at $\omega_{gc} = 10$ rad/sec.

Sol. given, transfer function $G(s)\,H(s) = \dfrac{Ks^2}{(1+0.25s)\,(1+0.025s)}$

$$\Rightarrow G(j\omega)\,H(j\omega) = \frac{K(j\omega)^2}{(1+0.025j\omega)\,(1+0.025j\omega)}$$

for $K=1 \Rightarrow G(j\omega)\,H(j\omega) = \dfrac{(j\omega)^2}{(1+0.25j\omega)(1+0.025j\omega)}$

Corner frequency

$$\omega_{c1} = \frac{1}{(1+0.25j\omega)} = \frac{1}{0.25} = 4 \text{ rad/sec}$$

$$\omega_{c2} = \frac{1}{(1+0.025j\omega)} = \frac{1}{0.025} = 40 \text{ rad/sec}$$

Now, various terms are listed below in ascending order.

Term	Corner freq. in rad/sec.	Slope in dB	change in slope in dB.
$(j\omega)^2$	–	40	–
$\dfrac{1}{(1+0.25j\omega)}$	$\omega_{c1} = \dfrac{1}{0.25} = 4$	-20	$40-20 = 20$
$\dfrac{1}{(1+0.025j\omega)}$	$\omega_{c2} = \dfrac{1}{0.025} = 40$	-20	$+20-20 = 0.$

Magnitude plot :-

Assume $\omega_c = 1$, $\omega_h = 100$.

Magnitude A at $\omega = \omega_c$ $= 20 \log \omega^2$

$= 20 \log (1)$

$= 0\ dB.$

A at $\omega = \omega_{c_1}$ $= 20 \log \omega^2$

$= 20 \log (4)^2$

$= 24\ dB.$

A at $\omega = \omega_{c_2}$ $= \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] + A \text{ at } \omega = \omega_{c_1}$

$= 20 \times \log \left(\frac{40}{4}\right) + 24$

$= 20 + 24 = 44\ dB.$

A at $\omega = \omega_h$ $= \left(\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}}\right) + A \text{ at } \omega = \omega_{c_2}$

$= 0 \times \log\left(\frac{100}{40}\right) + 44$

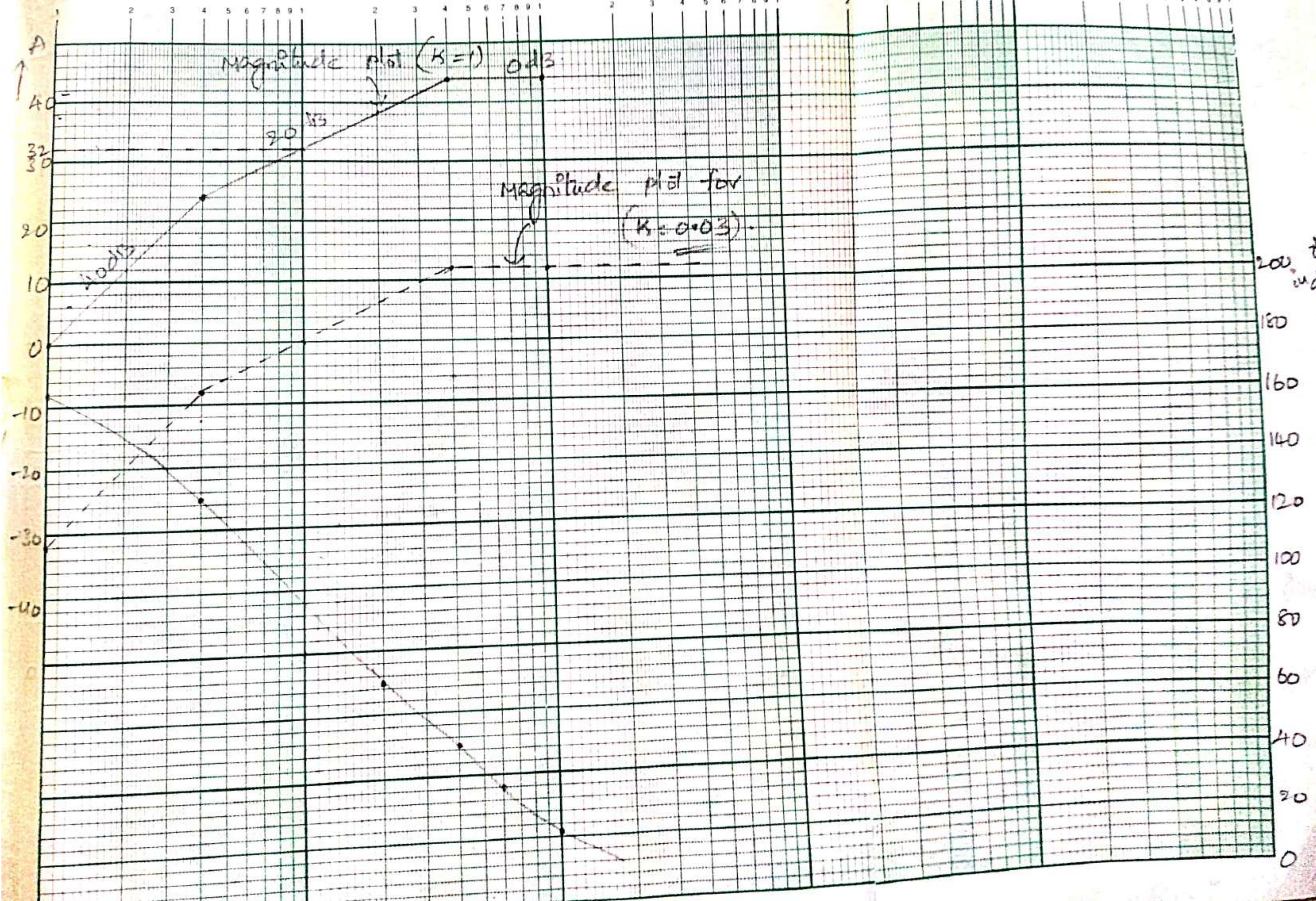
$= 44\ dB.$

Phase Plot :- $\phi = 180^\circ - \tan^{-1}(0.25\omega) - \tan^{-1}(0.025\omega)$

ω	$\tan^{-1}(0.25\omega)$	$\tan^{-1}(0.025\omega)$	ϕ in deg.
1	14.03	1.432	164.53°
4	45	5.71	130°
20	78.69	26.56	74.75°
40	84.28	45	50.72°
60	86.18	56.30	37.82°
100.	87.70	68.19	24.11°.

Now, mark the points regarding to the magnitudes for each ω values and phase angles for each ω values then connect the points there by we get, Magnitude Plot & phase plot.

② BODE PLOT FOR $G(S)H(S) = \dfrac{KS^2}{(1+0.25S)(1+0.025S)}$



given, to find K at $\omega_{gc} = 10$ rad/sec

$\Rightarrow 20\log K = -32$

$\Rightarrow \log K = \frac{-32}{20} = -1.6$

$\Rightarrow K = 0.0252 = 0.025$

$\simeq \underline{\underline{0.03}}$

$\therefore K = 0.03$ for $\omega_{gc} = 10$ rad/sec

$\omega_{gc} = 20$ rad/sec

$20\log K = -38$

$\log K = \frac{-38}{20} = -1.9$

$K = 10^{-1.9}$

$= \underline{\underline{0.0125}}$

3. Sketch the Bode plot for the following transfer function

$$G(s) = \frac{20}{s(1+3s)(1+4s)}$$, find gain margin, phase margin.

Sol. given T.F $\Rightarrow G(s) = \frac{20}{s(1+3s)(1+4s)}$.

to get the sinusoidal transfer function Replace $j\omega$ in s.

$$\Rightarrow G(j\omega) = \frac{20}{j\omega(1+3j\omega)(1+4j\omega)}$$

Now, the corner frequency for $(1+3j\omega) = \frac{1}{3} = 0.333$ rad/sec

" " " $(1+4j\omega) = \frac{1}{4} = 0.25$ rad/sec

Term	Corner frequency in rad/sec	Slope in dB	Change in slope in deg dB
$\frac{20}{j\omega}$	–	−20	–
$\frac{1}{(1+4j\omega)}$	$\omega_{c_1} = \frac{1}{4} = 0.25$	−20	$-20-20 = -40$
$\frac{1}{(1+3j\omega)}$	$\omega_{c_2} = \frac{1}{3} = 0.33$	−20	$-40-20 = -60$

Now, Assume lower corner frequency ω_l which is less than 0.25 and ω_h higher corner frequency which is greater than 0.33.

Now, let us Assume, $\omega_l = 0.15$, $\omega_h = 1$.

Now Calculate the Magnitude A for $\omega = \omega_l, \omega_{c_1}, \omega_{c_2}, \omega_h$.

A at $\omega = \omega_l$ $= 20\log\left(\frac{20}{\omega}\right)$

$= 20\log\left(\frac{20}{0.15}\right) = 42.5dB.$

A at $\omega = \omega_{c_1}$ $= 20\log\left(\frac{20}{\omega}\right)$

$= 20\log\left(\frac{20}{0.25}\right) = 38dB$

A at $\omega = \omega_{c_2}$ $= \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right] + A \text{ at } \omega = \omega_{c_1}$

$= \left[-40 \times \log\left(\frac{0.33}{0.25}\right)\right] + 38$

$= 33dB.$

A at $\omega = \omega_h$ $= \left[\text{slope from } \omega_{c_2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c_2}}\right] + A \text{ at } \omega = \omega_h$

$= -60 \times \log\left(\frac{1}{0.33}\right) + 33$

$= 4dB.$

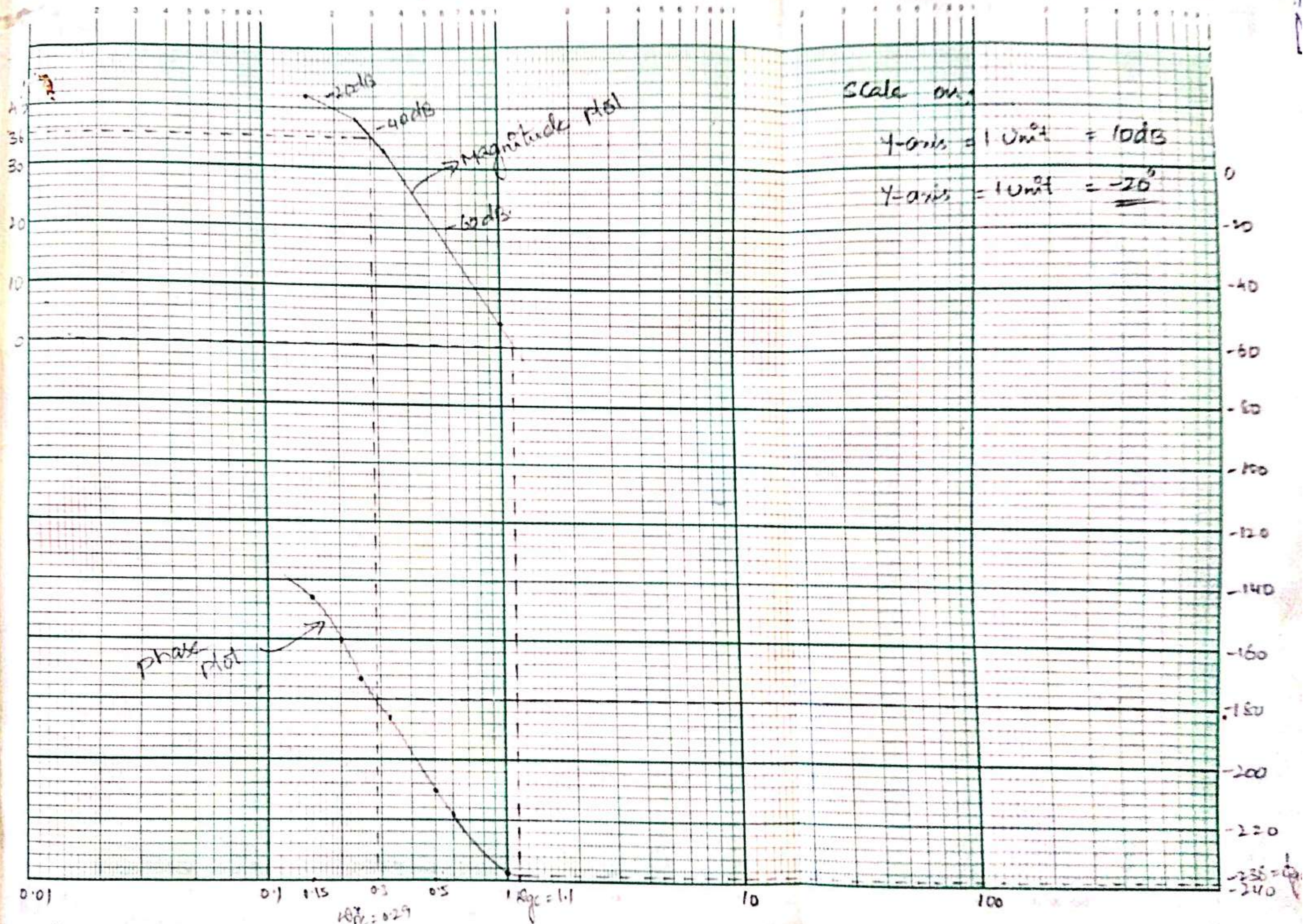
phase plot:

$\phi = -90^\circ - \tan^{-1}3\omega - \tan^{-1}4\omega.$

ω	$\tan^{-1}3\omega$	$\tan^{-1}4\omega$	$\phi = -90^\circ - \tan^{-1}3\omega - \tan^{-1}4\omega.$
0.15	24.22	30.96	$-145.18 \cong -145$
0.2	30.96	38.66	$-159.961 \cong -159 \cong -160$
0.25	36.86	45.0	$-171.82 \cong -172$
0.33	44.7	52.8	$-187.5 \cong -188$
0.5	56.30	6343	$-209.73 \cong -210$
0.6	60.14	67.38	$-218.32 \cong -218$
1	71.56	75.96	$-237.56 \cong -238$

Now, plot the Magnitude values in dB and phase angles in degrees on Semi-log graph paper for respective 'ω' values. we get, phase plot and the Magnitude plots.

③ BODE PLOT FOR $G(s) = \frac{20}{s(1+3s)(1+4s)}$



$\therefore$ The phase Margin $= \gamma = 180° + \phi_{gc}$

$$= 180° - 238°$$

$$= -58°$$

Now Gain Margin $= -20 \log |G(j\omega_{pc})|$ $\quad (\because \omega_{pc} = 0.29)$

$$= -20 \log \left(\frac{20}{0.29}\right)$$

$$= -36.7 \cong -36\,dB$$ $\therefore$ System is unstable

34. Sketch the Bode plot for the following transfer function and also obtain the phase margin and gain margin.

$$G(s) = \frac{30(1+0.1s)}{s(1+0.01s)(1+s)}$$

Sol: given, $G(s) = \dfrac{30(1+0.1s)}{s(1+0.01s)(1+s)}$.

$$G(j\omega) = \frac{30(1+0.1j\omega)}{j\omega(1+0.01j\omega)(1+j\omega)}.$$

The Corner frequency for the term $(1+0.1j\omega) = \omega = \frac{1}{0.1} = 10$

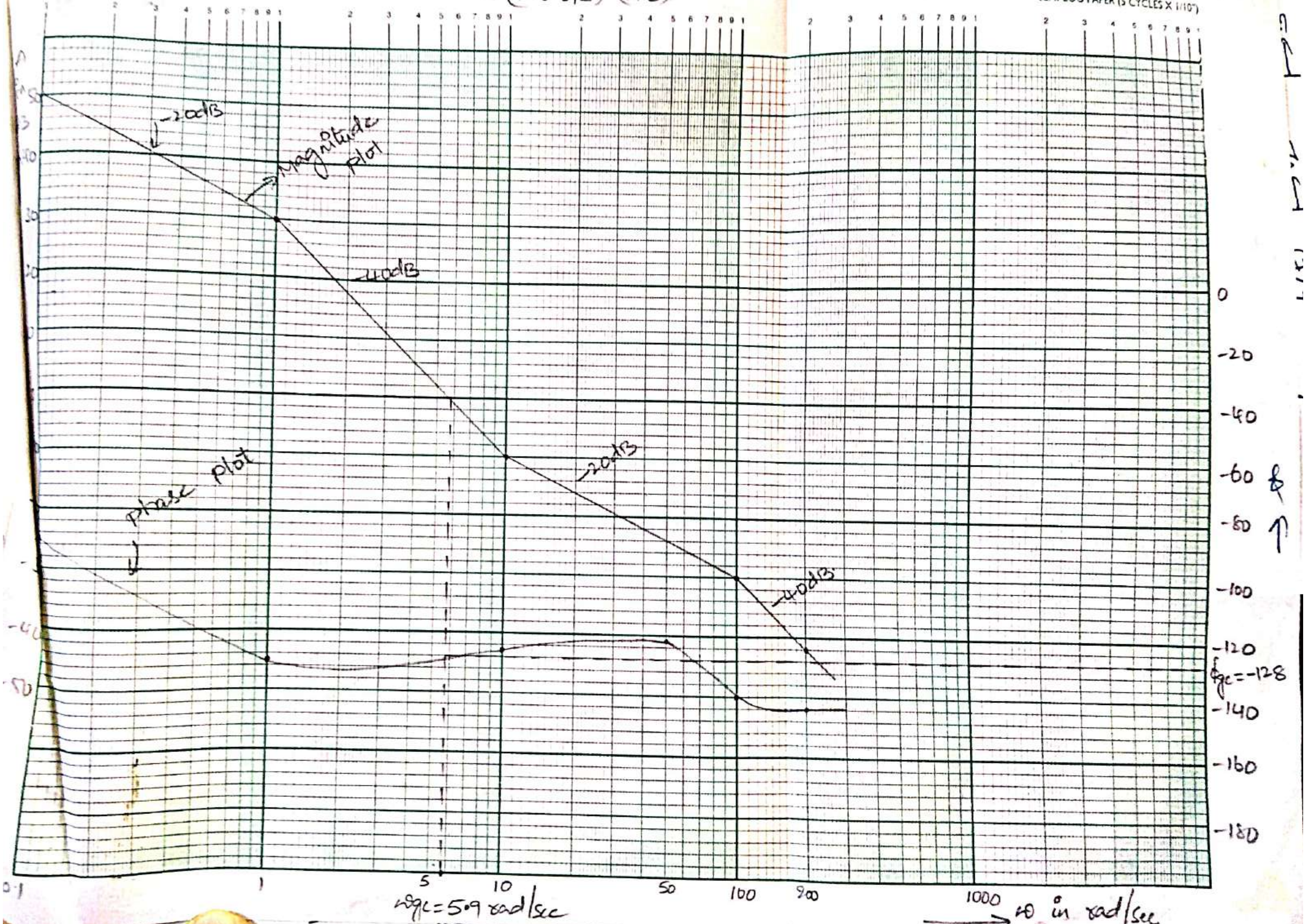
" " " " $(1+0.01j\omega) = \omega = \frac{1}{0.01} = 100$

" " " " $(1+j\omega) = \omega = \frac{1}{1} = 1$

Term	Corner frequency in rad/sec	Slope in dB.	Change in slope in dB.
$\frac{30}{j\omega}$	–	-20	–
$\frac{1}{(1+j\omega)}$	$\omega_{c_1} = \frac{1}{1} = 1$ rad/sec	-20	$-20-20 = -40$
$(1+0.1j\omega)$	$\omega_{c_2} = \frac{1}{0.1} = 10$ rad/sec	20	$-40+20 = -20$
$\frac{1}{(1+0.01j\omega)}$	$\omega_{c_3} = \frac{1}{0.01} = 100$ rad/sec	-20	$-20-20 = -40$

Now choose ω_l and ω_h as lower corner frequency and higher corner frequency.

Bode plot for $G(s) = \dfrac{30\,(1+0.1s)}{s\,(1+0.01s)\,(1+s)}$



$\omega_l = 0.1$, $\omega_h = 200.$

Now, Magnitudes for all the corner frequencies.

$\Rightarrow A_{at\ \omega=\omega_l} = 20\log\left(\frac{30}{\omega}\right)$

$= 20\log\left(\frac{30}{0.1}\right) = 49.5 \cong 50dB.$

$A\ at\ \omega=\omega_{c_1} = 20\log\left(\frac{30}{\omega}\right)$

$= 20\log\left(\frac{30}{1}\right) = 29.5 \cong 30dB.$

$A\ at\ \omega=\omega_{c_2} = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}}\right] + A\ at\ \omega=\omega_{c_1}$

$= -40\times\log\left(\frac{10}{1}\right) + 30$

$= -10dB.$

$A\ at\ \omega=\omega_{c_3} = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_{c_3} \times \log\frac{\omega_{c_3}}{\omega_{c_2}}\right] + A\ at\ \omega=\omega_{c_2}$

$= -20\times\log\left(\frac{100}{10}\right) - 10$

$= -20-10 = -30dB.$

$A\ at\ \omega=\omega_h \Rightarrow \left[\text{slope from } \omega_{c_3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c_3}}\right] + A\ at\ \omega=\omega_{c_3}$

$= -40\times\log\left(\frac{200}{100}\right) - 30$

$= -40\times \overset{0.301}{2} - 30 = \cancel{-80-30} = \cancel{-110dB}.$

$= -42dB.$

phase plot:

$\phi = -90^\circ + \tan^{-1}(0.1\omega) - \tan^{-1}(0.01\omega) - \tan^{-1}(1\omega).$

ω	$\tan^{-1}(0.1\omega)$	$\tan^{-1}(0.01\omega)$	$\tan^{-1}(1\omega)$	ϕ in deg
0.1				-95
1				$-129.8 \cong -130^\circ$
10				-135°
50				-126°
100				-140°
200				-156°.

∴ phase margin = $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ + (-128^\circ)$

$= 52^\circ$

the Gain margin $= \infty$ (infinity).

∴ even if we take $\omega_n = 10,000$ the value of ϕ if we substitute $\omega = 10,000$ in equation we get phase angle as -179.42 which is not equal to -180°.

* Requirement of phase margin and Gain margin for a System to be stable:

1) For a stable system, both gain margin and phase margin must be positive. Large values of gain margin and phase margin yield a relatively more stable system, but the system will be sluggish.

* Quadratic factor in the Numerator:

$$\Rightarrow G(s) = 1 + 2\xi\left(\frac{s}{\omega_n}\right) + \left(\frac{s}{\omega_n}\right)^2$$

$$\Rightarrow G(j\omega) = 1 + 2j\xi\left(\frac{\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2$$

$$= 1 + 2j\xi\left(\frac{\omega}{\omega_n}\right) - \left(\frac{\omega}{\omega_n}\right)^2$$

$$\Rightarrow \boxed{G(j\omega) = \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + 4\left(\xi\frac{\omega}{\omega_n}\right)^2} \angle \tan^{-1} \frac{2\xi\frac{\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}}$$

NOTE:- For the quadratic factor, the corner frequency will be the frequency ω_n.

* Quadratic factor in the Denominator:

$$G(s) = \frac{1}{1 + 2\xi\frac{s}{\omega_n} + \left(\frac{s}{\omega_n}\right)^2}$$

$$\Rightarrow G(j\omega) = \frac{1}{1 + 2\xi\frac{j\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2} = \frac{1}{1 + 2j\xi\frac{\omega}{\omega_n} - \left(\frac{\omega}{\omega_n}\right)^2}$$

NOTE :- For the quadratic factor in the denominator the the frequency ω will be ω_n.

5. Sketch the Bode plot for the following transfer function:

$$G(s) = \frac{40(1+s)}{(1+5s)(s^2+2s+4)}$$

$$\left((s^2+2s+4) \Rightarrow 4\left(\frac{s^2}{4}+\frac{1}{2}s+1\right) \Rightarrow 4(0.25s^2+s\cdot 0.5+1) \Rightarrow 4\left[s0.5+(1-0.25\omega^2)\right] \Rightarrow 4\left[0.5j\omega+(1-0.25\omega^2)\right] \right)$$

Sol. given, $G(s) = \dfrac{40(1+s)}{(1+5s)(s^2+2s+4)}$.

$$\Rightarrow G(j\omega) = \frac{40(1+j\omega)}{(1+5j\omega)\left[(j\omega)^2+2j\omega+4\right]}$$

$$= \frac{40(1+j\omega)}{(1+5j\omega)(-\omega^2+2j\omega+4)} = \frac{10(1+j\omega)}{(1+5j\omega)\left[(1-0.25\omega^2)+0.5s\right]}$$

Now, from the T.F $G(s)$ comparing the quadratic factor to the generalised second order system $s^2+2s\xi\omega_n+\omega_n^2 = s^2+2s+4$

$\omega_n^2 = 4$

$\omega_n = 2$

$2\omega_n\xi = 2$

$\xi = \frac{1}{2} = 0.5$.

Now, finding the corner frequencies for the 3 terms we get,

ω for term $(1+j\omega)$ $= \omega = \frac{1}{T} = 1$ rad/sec

ω for term $(1+5j\omega)$ $= \omega = \frac{1}{T} = \frac{1}{5} = 0.2$ rad/sec

the corner frequency for the quadratic term $\left[2j\omega+(4-\omega^2)\right]$

$\omega = \omega_n = 2$ rad/sec

Term	Corner frequency (rad/sec)	Slope (dB)	Change in slope (dB)
10	—	0	—
$\frac{1}{1+5j\omega}$	$\omega_{c_1} = \frac{1}{5} = 0.2$	-20	$0-20 = -20$
$1+j\omega$	$\omega_{c_2} = \frac{1}{1} = 1$	20	$-20+20 = 0$
$\frac{1}{2j\omega+4-\omega^2}$	$\omega_{c_3} = \omega_n = 2$	-40	$0-40 = -40$.

Now, choose lower corner frequency ω_l which is $\omega_l < \omega_{c_1}$ and higher corner frequency ω_h which should be $\omega_h > \omega_{c_3}$

let us assume $\omega_l = 0.1$ and $\omega_h = 10$.

Magnitude plot :-

Calculate the magnitude A for each corner frequency including ω_l and ω_h.

$$A \text{ at } \omega = \omega_l = 20 \log (10)$$
$$= \cancel{32\,dB}\ 20dB$$

$$A \text{ at } \omega = \omega_{c_1} = 20 \log (\cancel{40})(10)$$
$$= \cancel{32\,dB}\ 20dB.$$

$$A \text{ at } \omega = \omega_{c_2} = \left[\text{slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] + A_{\omega = \omega_{c_1}}$$
$$= -20 \times \log\left(\frac{1}{0.2}\right) + \cancel{3}20$$
$$= \cancel{18\,dB}.\ 6\,dB.$$

$$A \text{ at } \omega = \omega_{c_3} = \left[\text{slope from } \omega_{c_2} \text{ to } \omega_{c_3} \times \log \frac{\omega_{c_3}}{\omega_{c_2}}\right] + A_{\omega = \omega_{c_2}}$$
$$= 0 \times \log\left(\frac{2}{1}\right) + \cancel{18}\ 6$$
$$= \cancel{18\,dB}.\ 6\,dB.$$

$$A \text{ at } \omega = \omega_h = \left[\text{slope from } \omega_{c_3} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_3}}\right] + A_{\omega = \omega_{c_3}}$$
$$= -40 \times \log\left(\frac{10}{2}\right) + \cancel{18}\ 6$$
$$= -\cancel{9.95} \cong \cancel{-10\,dB}.\ -22\,dB.$$

Now, plot a graph on semi-log graph sheet to the corner frequencies (Vs) magnitude (A) in dB to obtain magnitude plot
(ω)

Phase plot :- $\phi = \tan^{-1}(\omega) - \tan^{-1}(5\omega) - \tan^{-1}\left(0.5\omega / 1-0.25\omega^2\right)$

For the quadratic function $\left[0.5j\omega + (1-0.25\omega^2)\right]$ the angle will be,

$$\boxed{\begin{aligned} &\tan^{-1}\frac{0.5\omega}{1-0.25\omega^2} \quad \text{for } \omega \le \omega_n. \\ &\tan^{-1}\frac{0.5\omega}{1-0.25\omega^2} + 180^\circ \quad \text{for } \omega > \omega_n. \end{aligned}}$$

and $\left[\because \text{Here, } \omega_n = 2\right]$

ω	$\tan^{-1}(\omega)$	$\tan^{-1}(5\omega)$	$\tan^{-1}\left[0.5\omega / (1-0.25\omega^2)\right]$	ϕ in deg.
0.1	5.71	26.56	2.869	$-23.72 \cong -24$
1.	45	78.69	33.69	-67.38
2	63.43	84.28	90	$-110.85 \cong -111$
4	75.96	87.13	$-33.69+180^\circ = 146.30$	-157°.
6	80.53	88.09	$-20.55+180^\circ = 159.44$	-167°
10.	84.28	88.85	$-11.76+180^\circ = 168.23$	$-172.8 \cong -173$.

Now plot all the magnitude values and phase values for the respective frequencies. we get a magnitude plot and the phase plot as shown in the graph sheet.

From the graph.

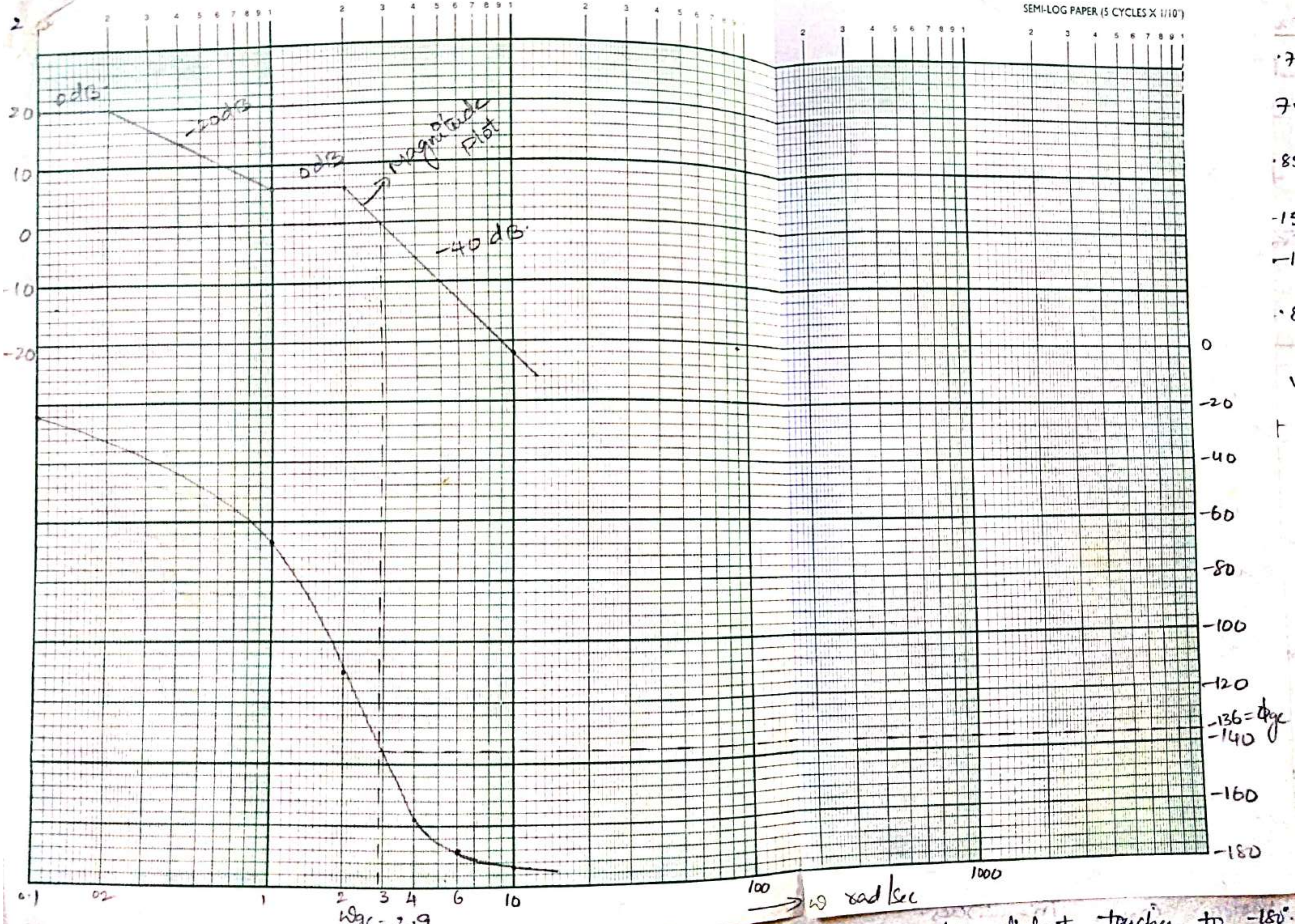
$$\phi_{gc} = -136^\circ \text{ (approximately)}.$$

$$\text{Now, Phase Margin} = \gamma = 180^\circ + \phi_{gc}$$
$$= 180^\circ - 136^\circ$$
$$= 44^\circ.$$

$$\text{Gain margin} = \infty \text{ (infinity)}.$$

Because the phase plot did not touches to -180°. So, there by no phase cross over frequency (ω_{pc}). $\therefore$ Gain Margin is ∞. (infinity).

⑤ BODE PLOT FOR $G(s) = \dfrac{40(1+s)}{(1+5s)(s^2+2s+4)}$



6. Sketch the Bode plot for the following transfer function

$$G(s) = \frac{10(1+s)e^{-0.1s}}{s(1+0.2s)}$$

Sol: given, $$G(s) = \frac{10(1+s)e^{-0.1s}}{s(1+0.2s)}$$

$$\Rightarrow G(j\omega) = \frac{10(1+j\omega)e^{-0.1j\omega}}{j\omega(1+0.2j\omega)}.$$

As there exists two time constant terms So, two Corner frequencies will obtained.

ω for term $(1+j\omega) = \frac{1}{T} = \frac{1}{1} = 1$ rad/sec

ω for term $(1+0.2j\omega) = \frac{1}{T} = \frac{1}{0.2} = 5$ rad/sec

Now Determine the slope and change in slope for respective terms. as shown below.

Term	Corner frequency in rad/sec	slope (dB)	change in slope in dB.
$\frac{10}{j\omega}$		-20	
$(1+j\omega)$	$\omega_{c_1} = \frac{1}{1} = 1$ rad/sec	20	$-20+20 = 0$
$\frac{1}{1+0.2j\omega}$	$\omega_{c_2} = \frac{1}{0.2} = 5$ rad/sec	-20	$0 - 20 = -20$

Now, Assume ω_l lower Corner frequency which will be $\omega_l < \omega_{c_1}$ and the Higher Corner frequency ω_h which should be $\omega_h > \omega_{c_2}$

let us Assume $\omega_l = 0.1$, $\omega_h = 100$. 100 rad/sec

Now Calculate the Magnitude values to plot the Magnitude Plot for various Corner frequencies.

A at $\omega = \omega_l \;=\; 20 \log |G(j\omega)|$

$$= 20 \log \left(\frac{10}{\omega}\right)$$

$$= 20 \log \left(\frac{10}{0.1}\right) = 20 \times 2 = 40 dB.$$

A at $\omega = \omega_{c_1} \;=\; 20 \log \left(\frac{10}{\omega}\right)$

$$= 20 \log \left(\frac{10}{1}\right) = 20 dB.$$

A at $\omega = \omega_{c_2} \;=\; \left[\text{Slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log \frac{\omega_{c_2}}{\omega_{c_1}}\right] + A \text{ at } \omega = \omega_{c_1}$

$$= 0 \times \log \left(\frac{5}{1}\right) + 20$$

$$= 20 dB.$$

A at $\omega = \omega_h \;=\; \left[\text{Slope from } \omega_{c_2} \text{ to } \omega_h \times \log \frac{\omega_h}{\omega_{c_2}}\right] + A_{\omega = \omega_{c_2}}$

$$= -20 \times \log \left(\frac{100}{5}\right) + 20$$

$$= -20 \times \log \left(\frac{100}{5}\right) + 20$$

$$= -6 dB.$$

Phase plot :-

$$e^{-i\theta} = \cos(-\theta) + j \sin(-\theta).$$

Now, the phase angle $\phi = \tan^{-1} \left[\frac{\sin(-\theta)}{\cos(-\theta)}\right]$

$$= \tan^{-1} [\tan(-\theta)]$$

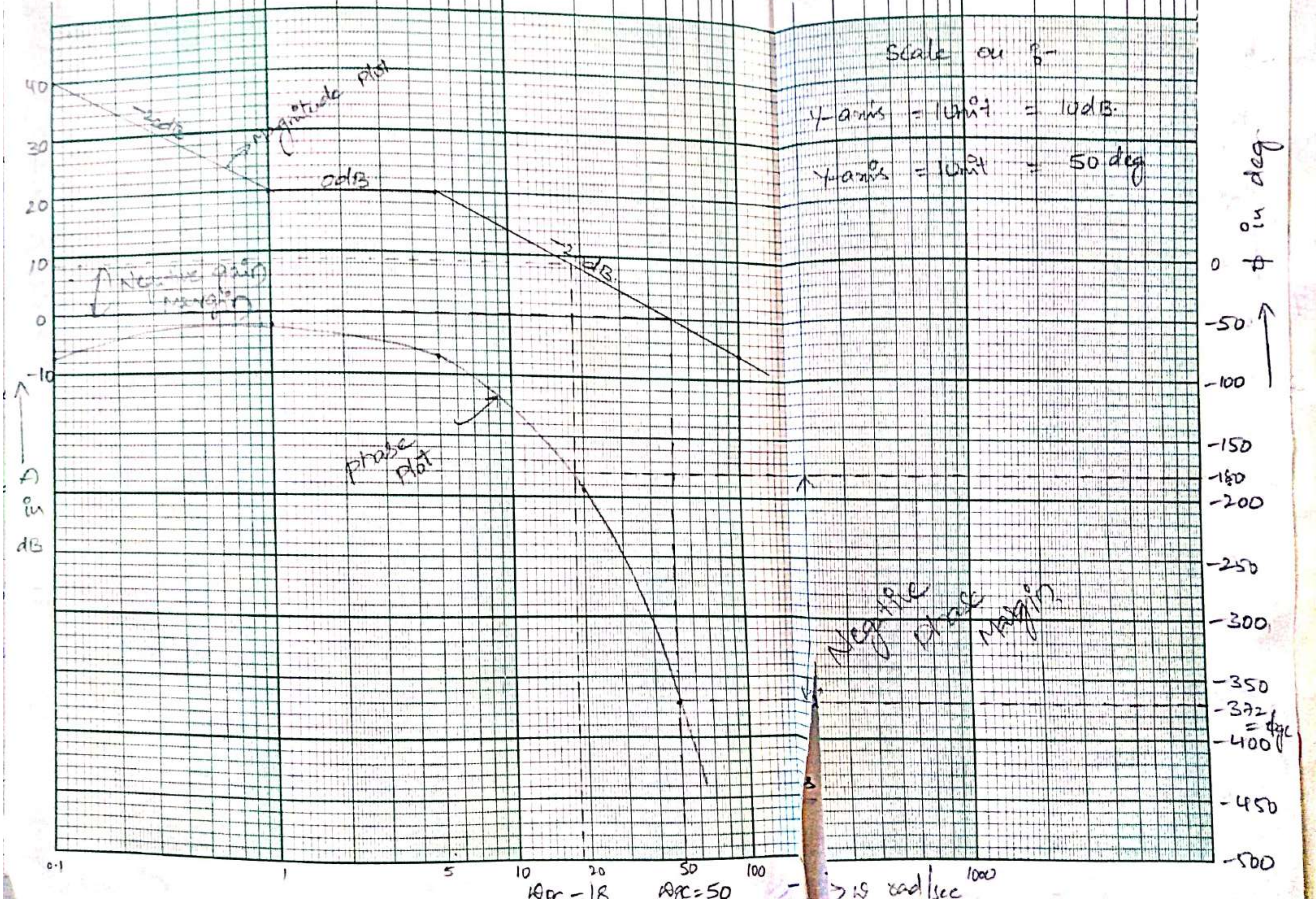
$$= -\theta \text{ radians.}$$

Similarly, $e^{-0.7j\omega}$ the angle must be -0.7ω radians.

Now, Converting Radians to degrees, $\Rightarrow \boxed{1 \text{ rad} \rightarrow \frac{180}{\pi} \text{ deg}}$

$$\Rightarrow -0.7j\omega \text{ rad} = -0.7\omega \times \frac{180}{\pi} \text{ deg} = \phi$$

6. Bode plot for $G(s) = \dfrac{10(1+s)e^{-0.1s}}{s(1+0.2s)}$.



Now, $\phi = -90^\circ - 0.1\omega \times \frac{180}{\pi} + \tan^{-1}(\omega) - \tan^{-1}(0.2\omega)$.

$$= -0.1\omega \times \frac{180}{\pi} - 90^\circ - \tan^{-1}(0.2\omega) + \tan^{-1}(\omega).$$

ω	$-0.1\omega \times \frac{180}{\pi}$	$\tan^{-1}(0.2\omega)$	$\tan^{-1}(1.\omega)$	ϕ in deg
0.1	-0.572	1.145	5.71	-86°
1	-5.72	11.30	45	-62°
5	-28.64	45	78.69	-85°.
20	-114.59	75.96	87.13	-193.42°
50	-286.47	84.28	88.8	-371.95°
100	-572.95	87.13	89.42	-660.61°.

Now, Mark the points of magnitude and phase angles for each and every corner frequencies. The plots we get is called magnitude plot and phase plot.

7. $G(s) = \frac{K}{s(s+2)(s+10)}$.

Sol. given, $G(s) = \frac{K}{s(s+2)(s+10)}$.

$$G(j\omega) = \frac{K}{s(s+2)(s+10)} = \frac{K}{j\omega(2+j\omega)(10+j\omega)}.$$

$$= \frac{K}{j\omega\, 2.(1+0.5j\omega)10(1+0.1j\omega)}$$

$$= \frac{0.05K}{j\omega(1+0.5j\omega)(1+0.1j\omega)}.$$

Corner frequency for term $(1+0.5j\omega) = \frac{1}{T} = \frac{1}{0.5} = 2$ rad/sec

Corner frequency for term $(1+0.1j\omega) = \frac{1}{T} = \frac{1}{0.1} = 10$ rad/sec

Now, assuming ω_l and ω_h.

let us assume, $\omega_h = 100$, $\omega_l = 1$.

(Term)	Corner frequency	slope	change in slope
$\frac{0.05}{j\omega}$	—	-20	—
$\frac{1}{1+0.5j\omega}$	$\omega_{c_1} = \frac{1}{0.5} = 2$ rad/sec	-20	$-20-20 = -40$
$\frac{1}{1+0.1j\omega}$	$\omega_{c_2} = \frac{1}{0.1} = 10$ rad/sec	-20	$-40-20 = -60$

Now, Magnitudes at each every frequencies including ω_l and ω_h.

$$A \text{ at } \omega=\omega_l = 20\log\left(\frac{0.05}{\omega}\right)$$

$$= 20\log\left(\frac{0.05}{1}\right) = -26\,\text{dB}.$$

$$A \text{ at } \omega=\omega_{c_1} = 20\log\left(\frac{0.05}{2}\right) = -32\,\text{dB}.$$

$$A \text{ at } \omega=\omega_{c_2} = \text{Slope from } \omega_{c_1} \text{ to } \omega_{c_2} \times \log\frac{\omega_{c_2}}{\omega_{c_1}} + A_{\omega=\omega_{c_1}}$$

$$= -40\times\log\left(\frac{10}{2}\right) + (-32)$$

$$= -58\,\text{dB}.$$

$$A \text{ at } \omega=\omega_h = \left[\text{Slope from } \omega_{c_2} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c_2}}\right] + A_{\text{at } \omega=\omega_{c_2}}$$

$$= -60\times\log\left(\frac{100}{10}\right) - 58$$

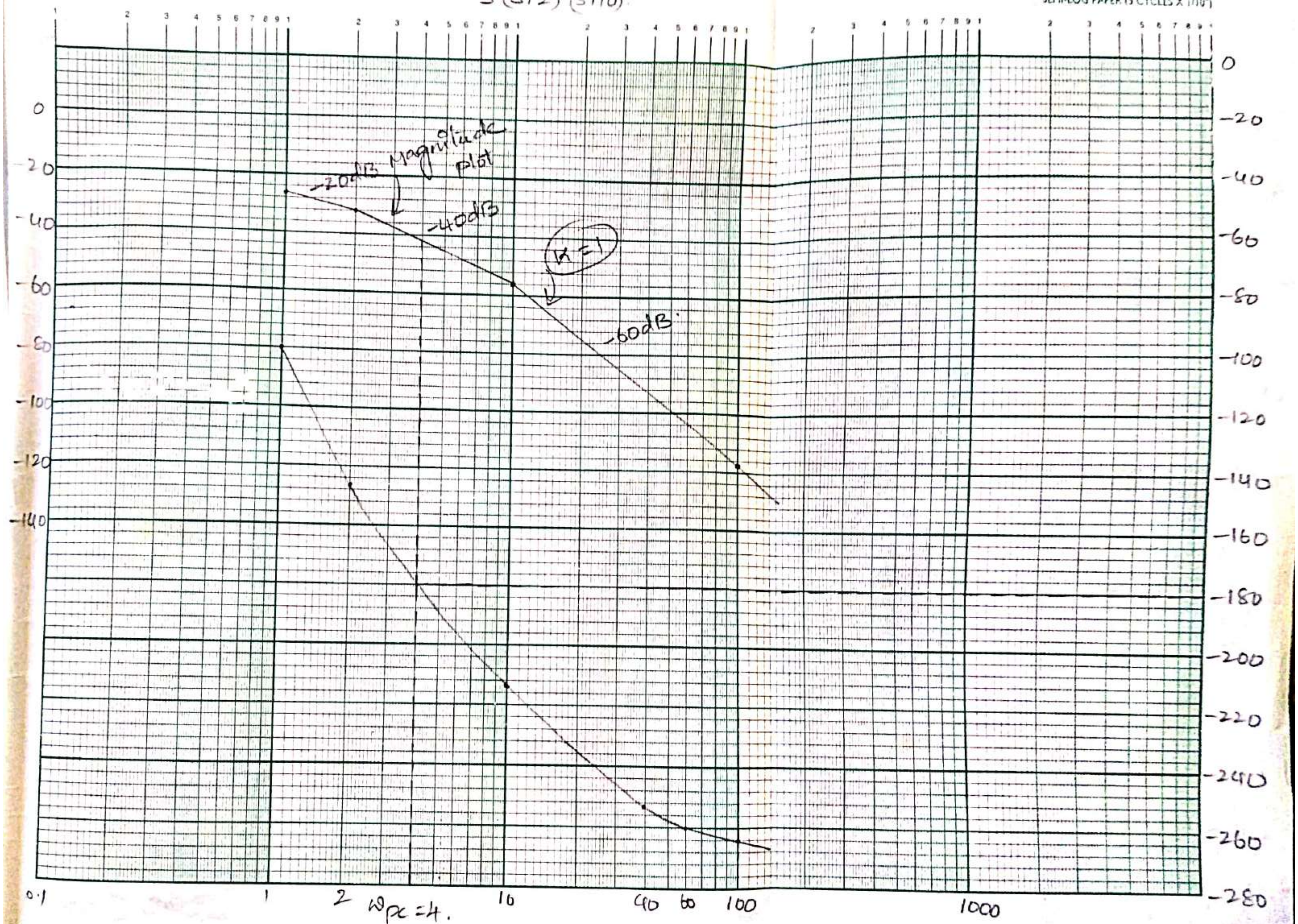
$$= -118\,\text{dB}.$$

Phase plot:-

$$\phi = -90° - \tan^{-1}(0.1\omega) - \tan^{-1}(0.5\omega).$$

ω	$\tan^{-1}(0.1\omega)$	$\tan^{-1}(0.5\omega)$	ϕ in deg
1	5.71	2.86	$-98.57 \cong -99$
2	11.30	45	-146.30
10	45	78.69	$-213.69 \cong -214$
40	75.96	87.13	-253.09
60	80.53	88.09	$-258.62 \cong -259$
100	84.28	88.85	-263.13

⑦ Bode Plot for $G(s) = \dfrac{K}{s(s+2)(s+10)}$



Now, $\omega_{pc} = 4$ rad/sec from the graph.

$$\therefore \text{gain Margin} = k_g = -20\log|G(j\omega)|$$

$$= -20\log\left(\frac{0.05}{\omega_{pc}}\right)$$

$$= -20\log\left(\frac{0.05}{4}\right)$$

$$= 38.06 \cong 38 dB.$$

Now, $k_g = -20\log K$

$$\Rightarrow 38 = -20\log K.$$

$$\Rightarrow K = 10^{\frac{-38}{20}}$$

$$= 10^{-1.9} = \underline{0.012}$$

For $K=0.012$ the Magnitude at $\omega_c = 0.1$ is -44.5 dB. That means it doesn't touches zero decible at any point of corner frequency. Therefore, there is no ω_{gc} & ϕ_{gc}.

Phase margin $\boxed{\gamma = 180° + \phi_{gc}}$

(8) $G(s) = \dfrac{s^2(s+10)}{(s+5)^2(s+0.1)}$.

Sol given, $G(s) = \dfrac{s^2(s+10)}{(s+5)^2(s+0.1)}$.

$$= \frac{s^2(s+10)}{(s^2+10s+25)(s+0.1)}$$

Now Compare the quadratic equation to Second order quadratic equation,

$$s^2+10s+25 = s^2+2\zeta\omega_n s + \omega_n^2$$

$$\Rightarrow \omega_n^2 = 25 \Rightarrow \omega_n = 5 \text{ rad/sec}$$

$\Rightarrow 2\omega_n \xi = 10$

$\Rightarrow 2 \times 5 \xi = 10$

$\Rightarrow \xi = 1.$

Now, $$G(s) = \frac{s^2 (s+10)}{(s^2+10s+25)(s+0.1)}$$

$$= \frac{s^2 \, 10(s0.1+1)}{(s^2+10s+25)\, 0.1\, (1+10s)}$$

$$\Rightarrow G(s) = \frac{100s^2(1+0.1s)}{(s^2+10s+25)(1+10s)} = \frac{100s^2(1+0.1s)}{25(0.04s^2+0.4s+1)(1+10s)}$$

Substitute $s = j\omega$ we get, sinusoidal transfer function,

$$\Rightarrow G(j\omega) = \frac{100(j\omega)^2(1+0.1j\omega)}{25[(j\omega)^2 0.04 + 0.4j\omega+1](1+10j\omega)}$$

$$= \frac{4(j\omega)^2(1+0.1j\omega)}{[(1-0.04\omega^2)+j0.4\omega](1+10j\omega)}$$

Now, Corner frequency is for $(1+0.1j\omega) = \frac{1}{0.1} = 10$ rad/sec

" " $(1+10j\omega) = \frac{1}{10} = 0.1$ rad/sec

" $(1-0.04\omega^2+j0.4\omega) = \omega_{c_2} = \omega_n = 5$ rad/sec

Term	Corner frequency	Slope	Change in slope
$4(j\omega)^2$	–	40	–
$\frac{1}{(1+10j\omega)}$	$\omega_{c_1} = \frac{1}{10} = 0.1$	−20	$40-20 = 20$
$(1-0.04\omega^2+j0.4\omega)$	$\omega_{c_2} = \omega_n = 5$	−40	$20-40 = -20$
$(1+0.1j\omega)$	$\omega_{c_3} = \frac{1}{0.1} = 10$	20	$-20+20 = 0.$

Assume ω_l and ω_h. Now, let us assume $\omega_l = 0.01$

and ω_h = ~~100.~~ ~~50~~ 100.

Now, Magnitude A at every corner frequencies.

$$A_{\omega=\omega_{c1}} = 20\log(4\omega^2)$$

$$= 20\log\left[4\times(0.01)^2\right]$$

$$= -67.95 \simeq -68\,dB.$$

$$A_{\omega=\omega_{c4}} = 20\log(4\omega^2)$$

$$= 20\log\left[4(0.1^2)\right]$$

$$= -27.95 \simeq -28\,dB$$

$$A_{\omega=\omega_{c2}} = \left[\text{slope from } \omega_{c4} \text{ to } \omega_{c2} \times \log\frac{\omega_{c2}}{\omega_{c4}}\right] + A_{\omega=\omega_{c4}}$$

$$= 20\times\log\left(\frac{5}{0.1}\right) + (-28)$$

$$= 5.97 \simeq 6\,dB.$$

$$A_{\omega=\omega_{c3}} = \left[\text{slope from } \omega_{c2} \text{ to } \omega_{c3} \times \log\frac{\omega_{c3}}{\omega_{c2}}\right] + A_{\omega=\omega_{c2}}$$

$$= -20\times\log\left(\frac{10}{5}\right) + 6$$

$$= \cancel{6dB}.\ -0.02 \simeq 0\,dB.$$

$$A_{\omega=\omega_h} = \left[\text{slope from } \omega_{c3} \text{ to } \omega_h \times \log\frac{\omega_h}{\omega_{c3}}\right] + A_{\omega=\omega_{c3}}$$

$$= -\overset{0}{\cancel{20}}\times\log\left(\frac{100}{10}\right) + 0$$

$$= -\overset{0}{\cancel{20}}\times\log(10) + 0 = \cancel{26dB}.\ \cancel{-14dB}.\ 0dB.$$

Phase plot:

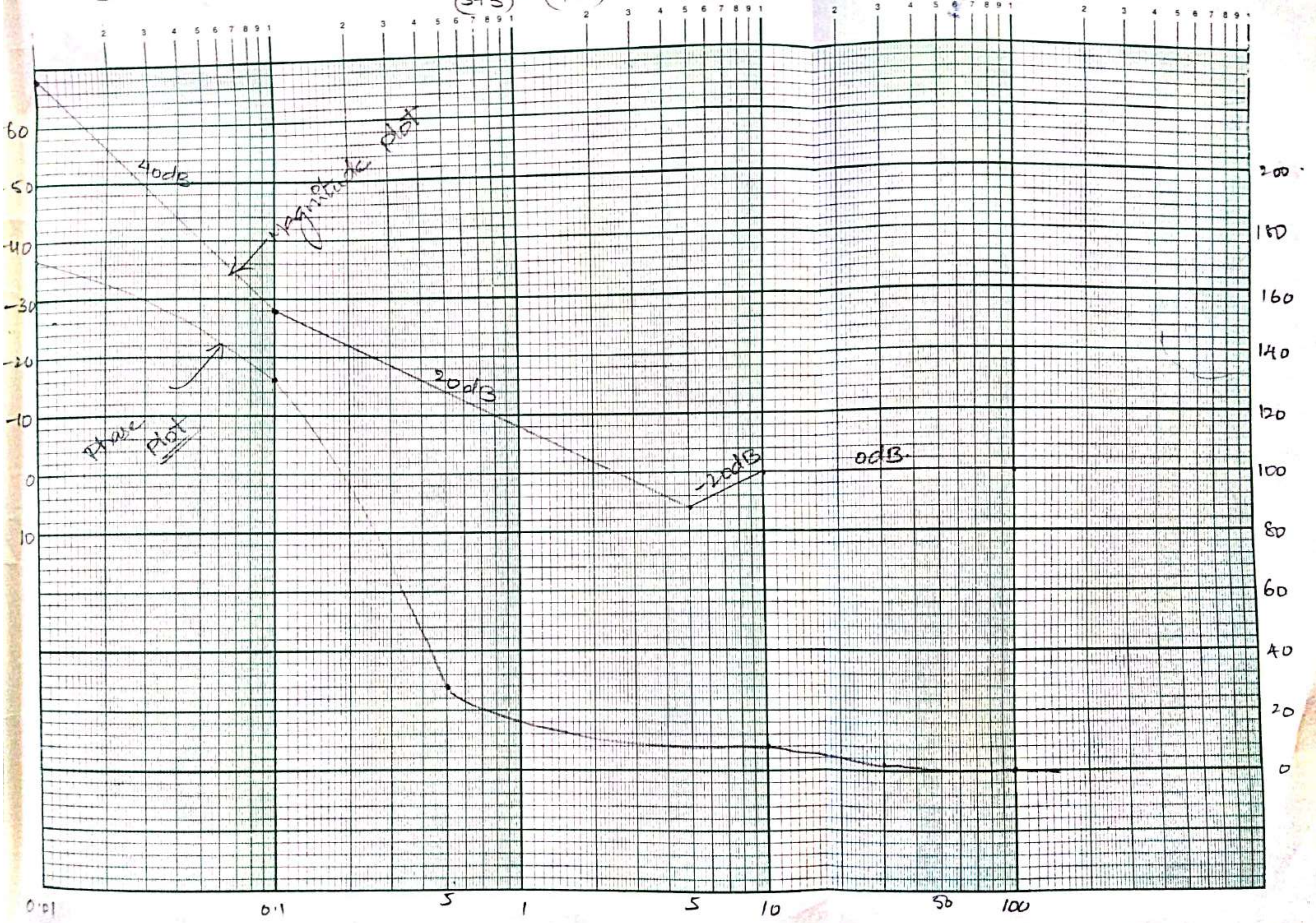
The phase angle for $(1-0.04\omega^2+0.4j\omega)$ will be,

$$\boxed{\tan^{-1}\left(\frac{0.4\omega}{1-0.04\omega^2}\right) \text{ for } \omega \le \omega_n \quad \text{and} \quad \tan^{-1}\left(\frac{0.4\omega}{1-0.04\omega^2}\right) + 180^\circ \text{ for } \omega > \omega_n.}$$

Now, $\phi = 180^\circ + \tan^{-1}(0.1\omega) - \tan^{-1}(10\omega) - \tan^{-1}\left[\frac{0.4\omega}{1-0.04\omega^2}\right]$.

ω	$\tan^{-1}(0.1\omega)$	$\tan^{-1}(10\omega)$	$\tan^{-1}\left(0.4\omega / 1-0.04\omega^2\right)$.	ϕ in deg
0.01	0.0572	5.71	0.229	174°
0.1	0.572	45	2.29	133.28°
5	26.56	88.85	90	27.71°
10	45	89.42	$-53.13+180^\circ = 126.86^\circ$	8.72°
50	78.69	89.88	$-11.42+180^\circ = 168.57^\circ$	0.24°
100	84.28	89.94	$-5.72+180^\circ = 174.27^\circ$	0.07°

(f) Bode plot for $G(s) = \dfrac{s^2 (s+10)}{(s+5)^2 (s+0.1)}$.



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Unit - 6

Stability Analysis

* Polar plot :-

The Polar plot of a sinusoidal transfer function $G(j\omega)$ is a plot of the Magnitude of $G(j\omega)$ Versus the phase angle of $G(j\omega)$ on polar Co-ordinates as ω is varied from zero to infinity. The polar plot is the locus of the vectors $|G(j\omega)| \angle G(j\omega)$ from 0 to ∞. The polar plot is also called as Nyquist plot.

The polar plot is usually plotted on a polar graph sheet. The polar graph sheet has concentric circles and Radial lines. The Radial lines Represent the phase angles and the circles represent the Magnitudes. Each point on the polar graph has Magnitude. In polar graph sheet the phase angle and positive phase angle is measured in anti-clock wise direction from the Reference axis (0°). Similarly, the Negative angle is measured in clock wise Direction from Reference axis (0°).

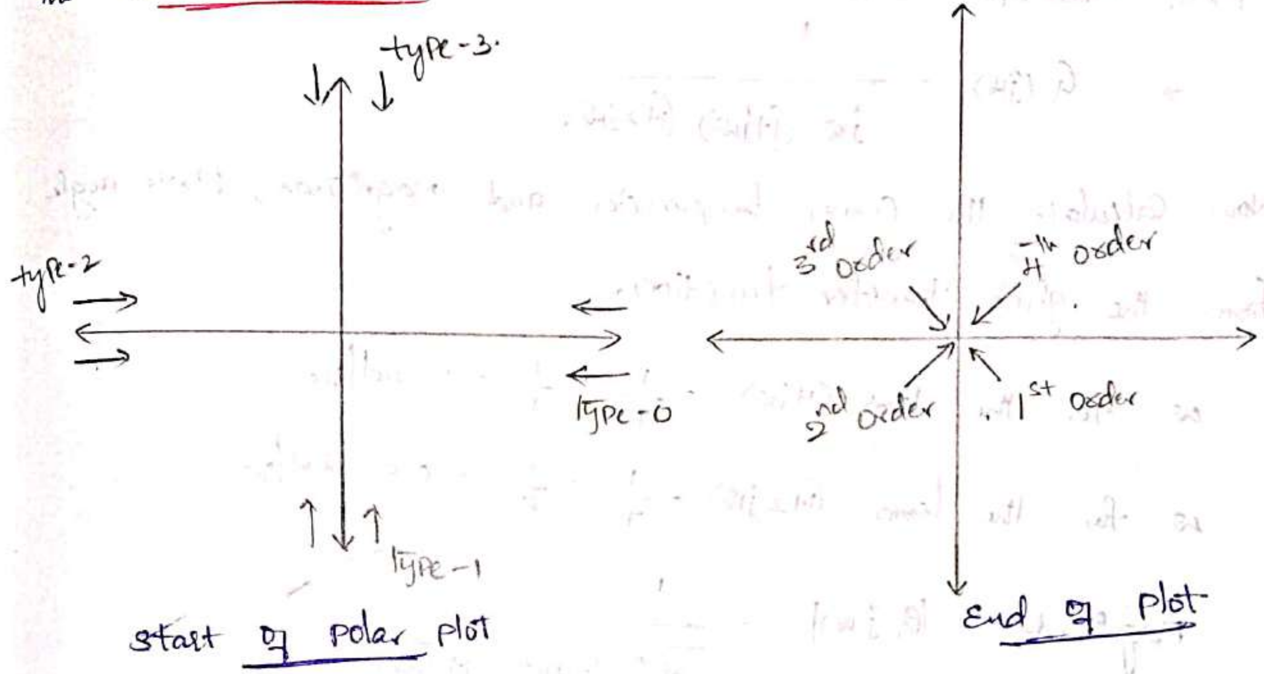
Alternatively, if $G(j\omega)$ can be expressed in the form of Rectangular co-ordinates as,

$$G(j\omega) = G_R(j\omega) + G_I(j\omega)$$

where, $G_R(j\omega)$ = Real part of $G(j\omega)$

$G_I(j\omega)$ = Imaginary part of $G(j\omega)$

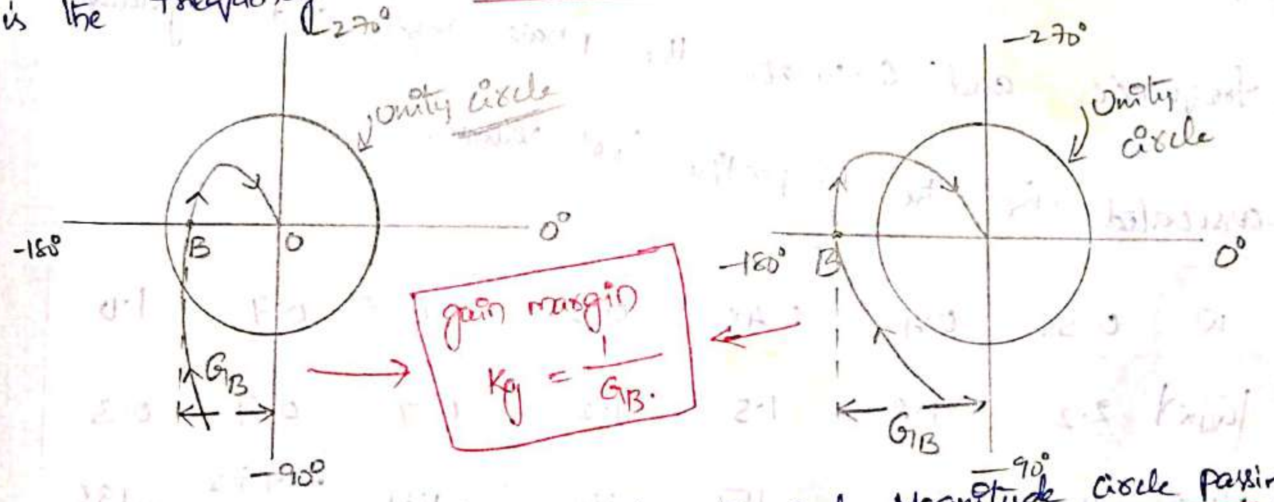
For minimum transfer function with only poles, type of the system determines at what quadrant the polar graph starts and the order of the system determines at what quadrant the polar plot ends.



* Determination of gain & phase Margin :-

Now let us see about the gain margin.

* The gain margin is defined as the inverse of magnitude of the $G(j\omega)$ at phase cross over frequency. The phase cross over freq. is the frequency at which the phase of $G(j\omega)$ is $-180°$.



let polar plot cut $180°$ axis at point B and magnitude circle passing through point B be G_B. Now gain margin $k_g = \frac{1}{G_B}$. If point 'B' lies with in the unity circle then gain margin is positive, otherwise the gain margin is Negative.

1. Sketch the polar plot for the transfer function $G(s) = \frac{1}{s(1+s)(1+2s)}$ and determine phase margin, gain margin.

Sol: given T.F $= G(s) = \frac{1}{s(1+s)(1+2s)}$

Now, Substitute $s = j\omega$

$$\Rightarrow G(j\omega) = \frac{1}{j\omega(1+j\omega)(1+2j\omega)}$$

Now, Calculate the corner frequencies and Magnitude, phase angle from the given transfer function,

ω for the term $(1+j\omega) = \frac{1}{T} = \frac{1}{1} = 1$ rad/sec

ω for the term $(1+2j\omega) = \frac{1}{T} = \frac{1}{2} = 0.5$ rad/sec

$$\text{Magnitude } |G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+\omega^2}\sqrt{1+(2\omega)^2}}$$

$$= \frac{1}{\omega\sqrt{1+\omega^2}\sqrt{1+4\omega^2}}$$

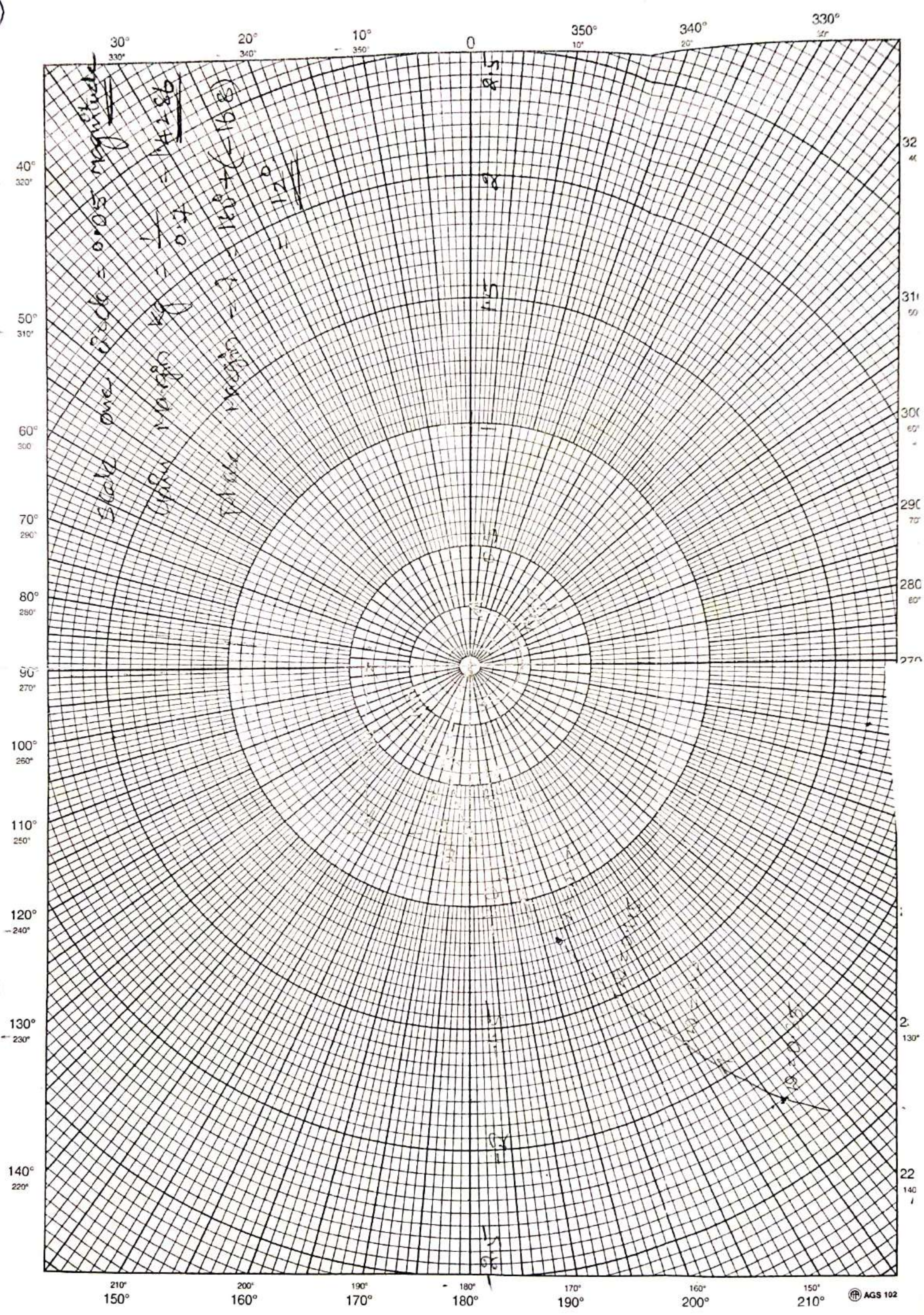
Phase angle of $G(j\omega) = \angle G(j\omega) = -90° - \tan^{-1}(\omega) - \tan^{-1}(2\omega)$

Now assume some frequency values around the corner frequencies and calculate the phase angle and magnitudes associated for the respective 'ω' values.

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$\lvert G(j\omega)\rvert$	2.2	1.8	1.5	1.2	0.9	0.7	0.3
$\angle G(j\omega)$	−144	−150	−156	−162	−171	−179.5 = −180	−198

Now Convert the polar form values to Rectangular form and then Mark the points on the polar & Rectangular graphs

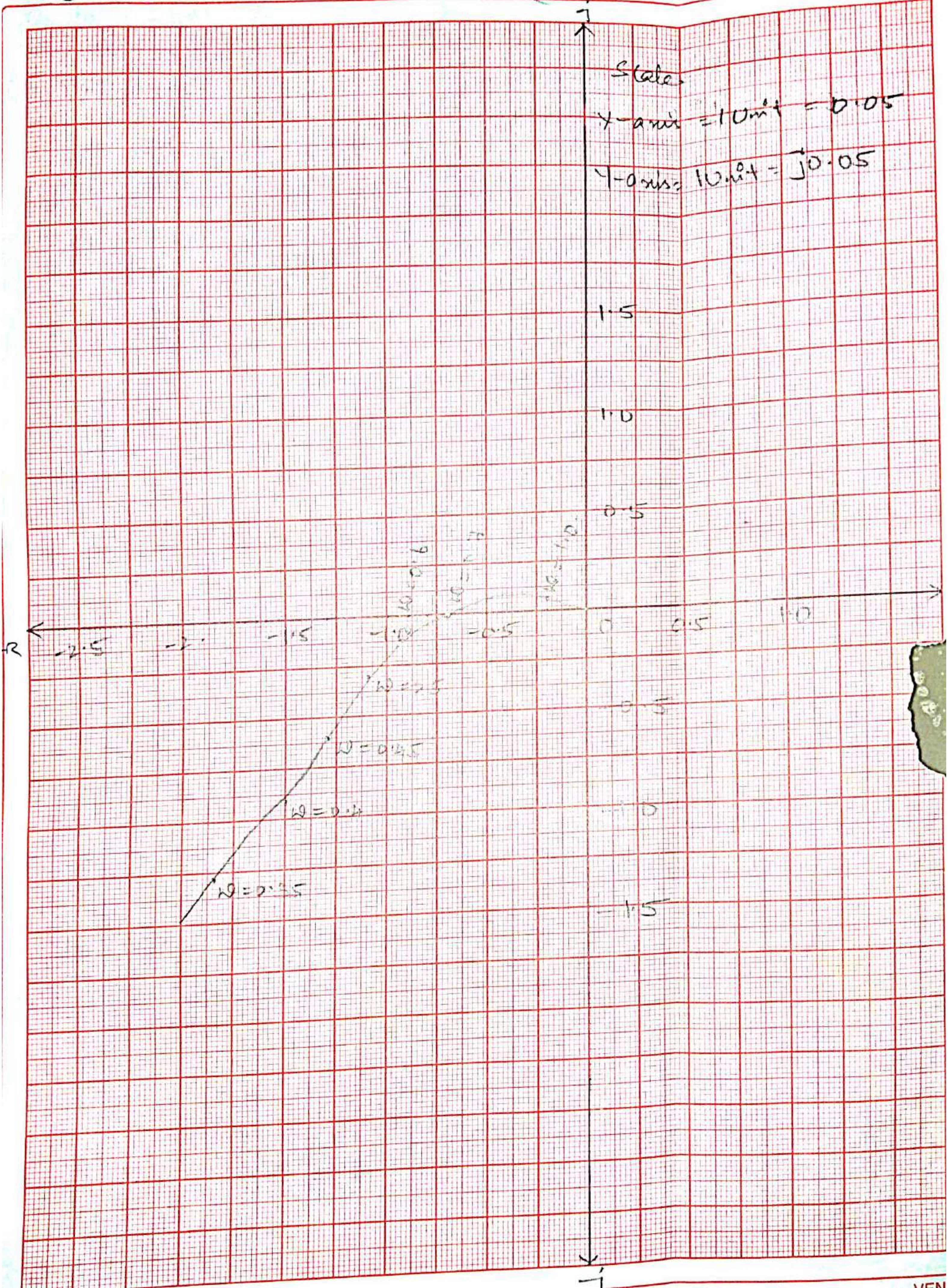
1



Scale one scale = 0.05 magnitude
Gain Margin Kg = 1/0.7 = 1.428
Phase Margin γ = 180 + (−168) = 12°
AGS 102

Page No.

1. Polar Plot on $G(s) = \frac{1}{s(1+s)(1+2s)}$.



respectively.

ω	0.35	0.4	0.45	0.5	0.6	0.7	1.0
$G_R(j\omega)$	-1.78	-1.56	-1.37	-1.14	-0.89	-0.7	-0.29
$G_I(j\omega)$	-1.29	-0.9	-0.61	-0.37	-0.14	0	0.09

$$\text{gain Margin} = \frac{1}{G_A} = \frac{1}{G_{1\omega A}}$$

$$G_A = 0.7$$

$$\therefore \text{gain Margin} = 1/G_A$$

$$k_g = \frac{1}{0.7} = 1.4286.$$

$180+19.8$

$\Rightarrow 378$

$\Rightarrow +18^{0}$

Similarly, $\gamma = 180^\circ + \phi_{gc}$

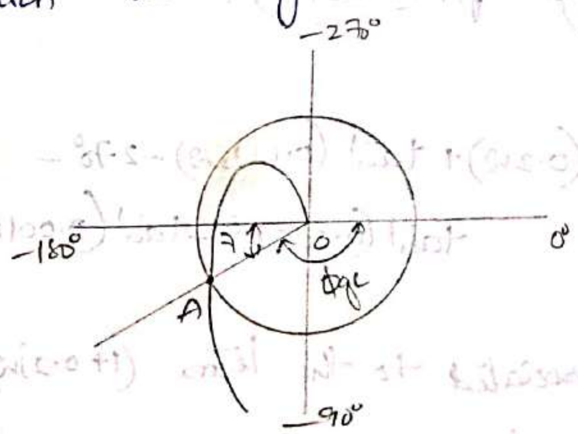
$$= 180^\circ - 168^\circ$$

$$= 12^\circ$$

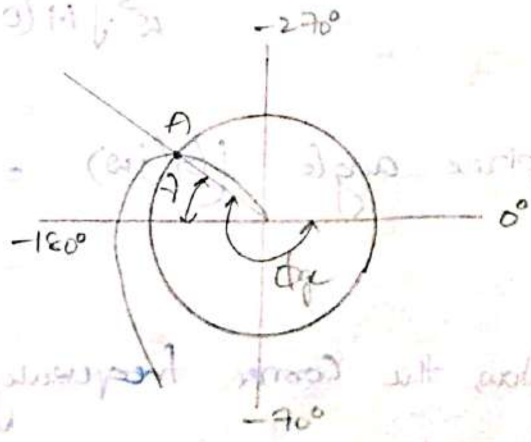
*** Determination of phase margin :-**

The phase margin is defined as, phase margin $\gamma = 180^\circ + \phi_{gc}$, where ϕ_{gc} is the phase angle of $G(j\omega)$ at gain cross-over frequency.

"The gain crossover frequency is the frequency at which the magnitude of $G(j\omega)$ is Unity".



Phase Margin $\gamma = 180^\circ + \phi_{gc}$



Phase Margin $= \gamma = 180^\circ + \phi_{gc}$

The phase margin is defined as, $\gamma = 180° + \phi_{gc}$

from the above two figs. the phase margin γ is given by $\angle AOP$. i.e, if $\angle AOP$ is below $-180°$ then phase margin is positive and if it is $\angle AOP$ is above $-180°$ axis then the phase margin is Negative.

2. Sketch the polar plot for the open loop transfer function

$$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$ and Determine phase margin.

Sol. given $$G(s) = \frac{(1+0.2s)(1+0.025s)}{s^3(1+0.005s)(1+0.001s)}$$

$$G(j\omega) = \frac{(1+0.2j\omega)(1+0.025j\omega)}{(j\omega)^3(1+0.005j\omega)(1+0.001j\omega)}$$

Now calculate the magnitude and the phase angle for the given transfer function.

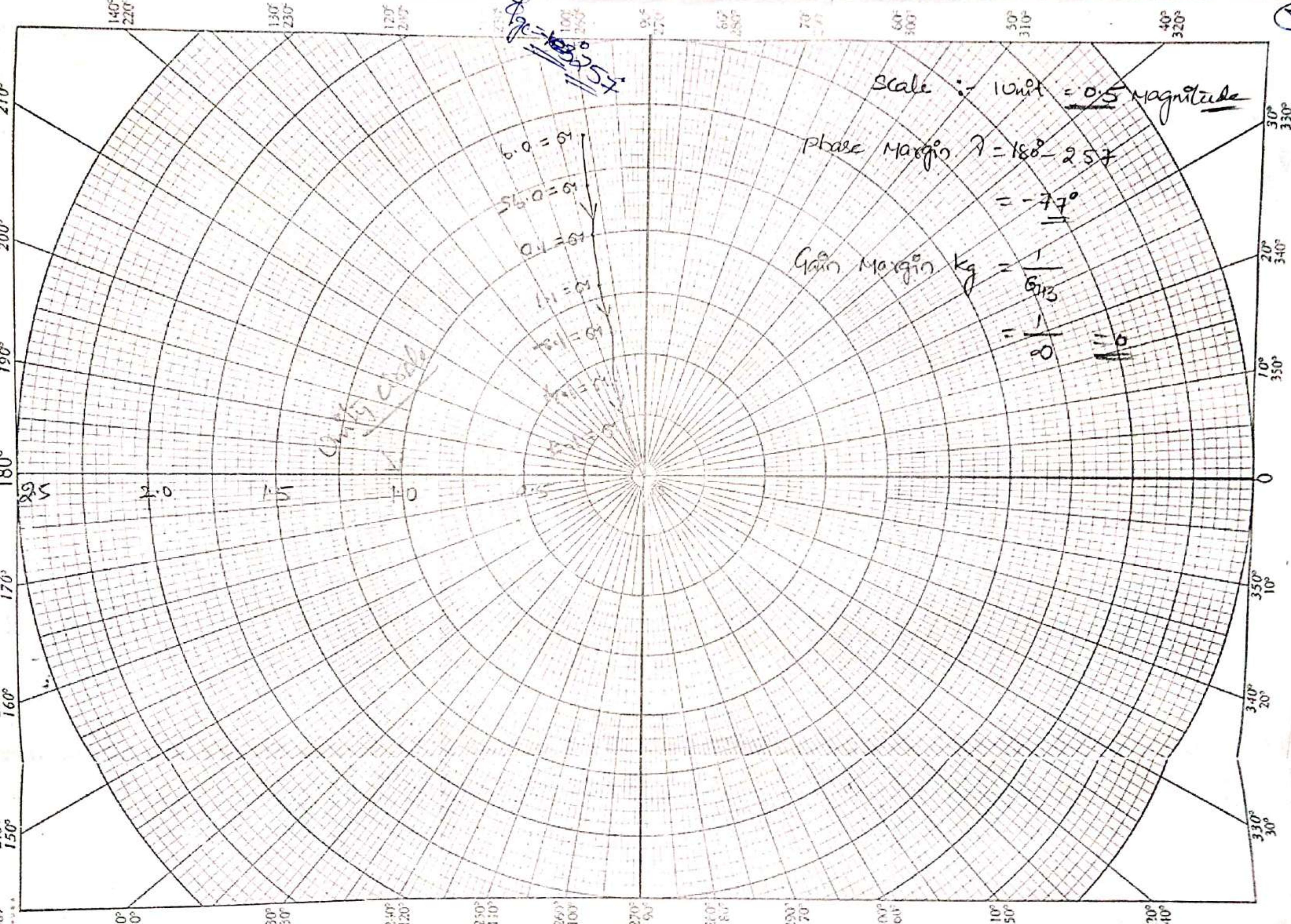
$$|G(j\omega)| = \frac{\sqrt{1+(0.2j\omega)^2}\sqrt{(1+(0.025\omega j)^2)}}{\sqrt{1+(0.005j\omega)^2}\sqrt{1+(0.001j\omega)^2}\cdot\omega^3}$$

$$= \frac{\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.025\omega)^2}}{\omega^3\sqrt{1+(0.005\omega)^2}\sqrt{1+(0.001\omega)^2}}$$

$$\frac{\sqrt{0+(\omega^3)^2}}{\sqrt{(\omega 3)^2}} = \omega^3$$

phase angle $$\angle G(j\omega) = \tan^{-1}(0.2\omega) + \tan^{-1}(0.025\omega) - 270° - \tan^{-1}(0.005\omega) - \tan^{-1}(0.001\omega).$$

Now, the corner frequency associated to the term $(1+0.2j\omega)$ is $\omega = \frac{1}{T} = \frac{1}{0.2} = 5$ rad/sec


Scale :- 1 Unit = 0.5 Magnitude
Phase Margin γ = 180° − 257
= −77°
Gain Margin Kg = 1/G_{1.3}
= 1/0 = 0
φgc = −180° − 257
ω = 0.9
ω = 0.95
ω = 1.0
ω = 1.1
ω = 1.2
ω = 1.3
Unity circle
0.5
1.0
1.5
2.0
2.5
180°
0
12-187

②

90

Scale on

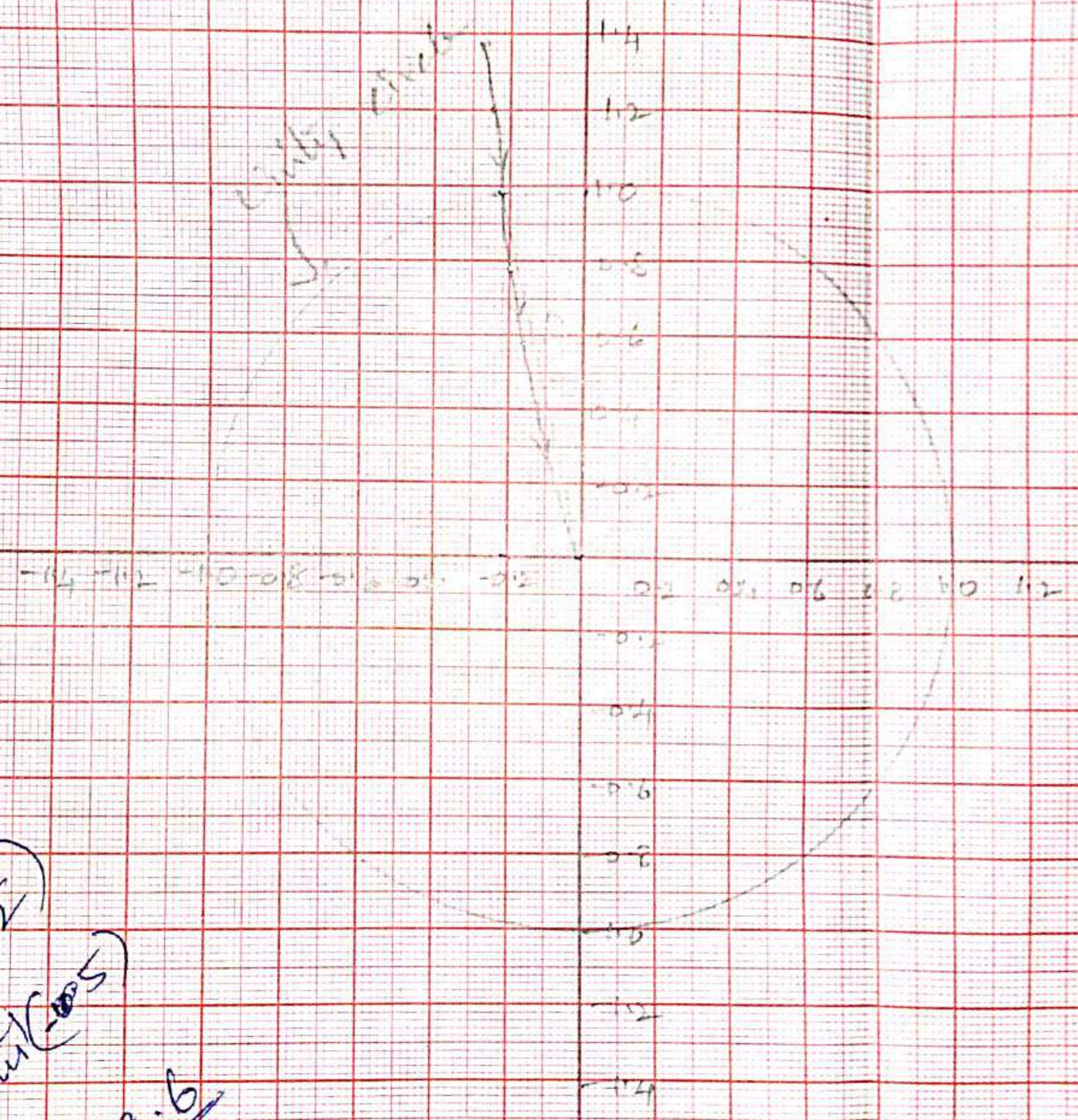
X-axis = 1cm = 0.2

Y-axis = 1cm = j0.2

Gain margin $kg = \frac{1}{OB} = \frac{1}{\infty} = 0$

180

0



$\tan^{-1}\left(\frac{1}{0.2}\right)$

$\tan^{-1}(.005)$

$\Rightarrow$

$\theta \Rightarrow -78.6$

270

ω for the term $(1+0.025j\omega) = \frac{1}{0.025}$

$= 40$ rad/sec

ω for the term $(1+0.005j\omega) = \frac{1}{T} = \frac{1}{0.005}$

$= 200$ rad/sec

ω for the term $(1+0.001j\omega) = \frac{1}{0.001} = 1000$ rad/sec

Now, calculate the magnitude and phase angle for the various frequencies and tabulate them as below.

ω	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$\lvert G(j\omega)\rvert$	1.4	1.2	1.0	0.8	0.6	0.4	0.2
$\angle G(j\omega)$	-259	-258	-257	-256	-255	-253	-249.

Now take a polar graph and plot the graph using the magnitude and phase angle values for the respective frequencies.

Now using the polar to Rectangular conversion, the polar co-ordinates listed in the above table are converted to the Rectangular co-ordinates as shown below.

ω	0.9	0.95	1.0	1.1	1.2	1.4	1.7
$G_R(j\omega)$	-0.27	-0.25	-0.22	-0.19	-0.16	-0.12	-0.07
$G_I(j\omega)$	1.37	1.17	0.97	0.78	-0.58	-0.38	0.19.

Now, phase Margin $\gamma = 180° + \phi_{gc}$

$= 180° - 257°$

$= -77°$

Gain Margin $K_g = \frac{1}{G_B}$

$= \frac{1}{\infty} = 0.$

3. Sketch the polar plot for the transfer function

$$G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$$ and Determine the gain margin and phase margin. Calculate the value of K at gain margin 18dB.

Sol given transfer function, $G(s) = \frac{K}{s(1+0.2s)(1+0.05s)}$

Change this into sinusoidal transfer function.

$$G(j\omega) = \frac{K}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$$

Now polar plot is to be plotted by assuming the value of K=1 and then find the actual value of K.

Assume, $K=1 \Rightarrow G(j\omega) = \frac{1}{j\omega(1+0.2j\omega)(1+0.05j\omega)}$.

Corner frequency ω for the term $(1+0.2j\omega) = \frac{1}{T} = \frac{1}{0.2}$

$= 5$ rad/sec

Corner frequency ω for the term $(1+0.05j\omega) = \frac{1}{T} = \frac{1}{0.05}$

$= 20$ rad/sec

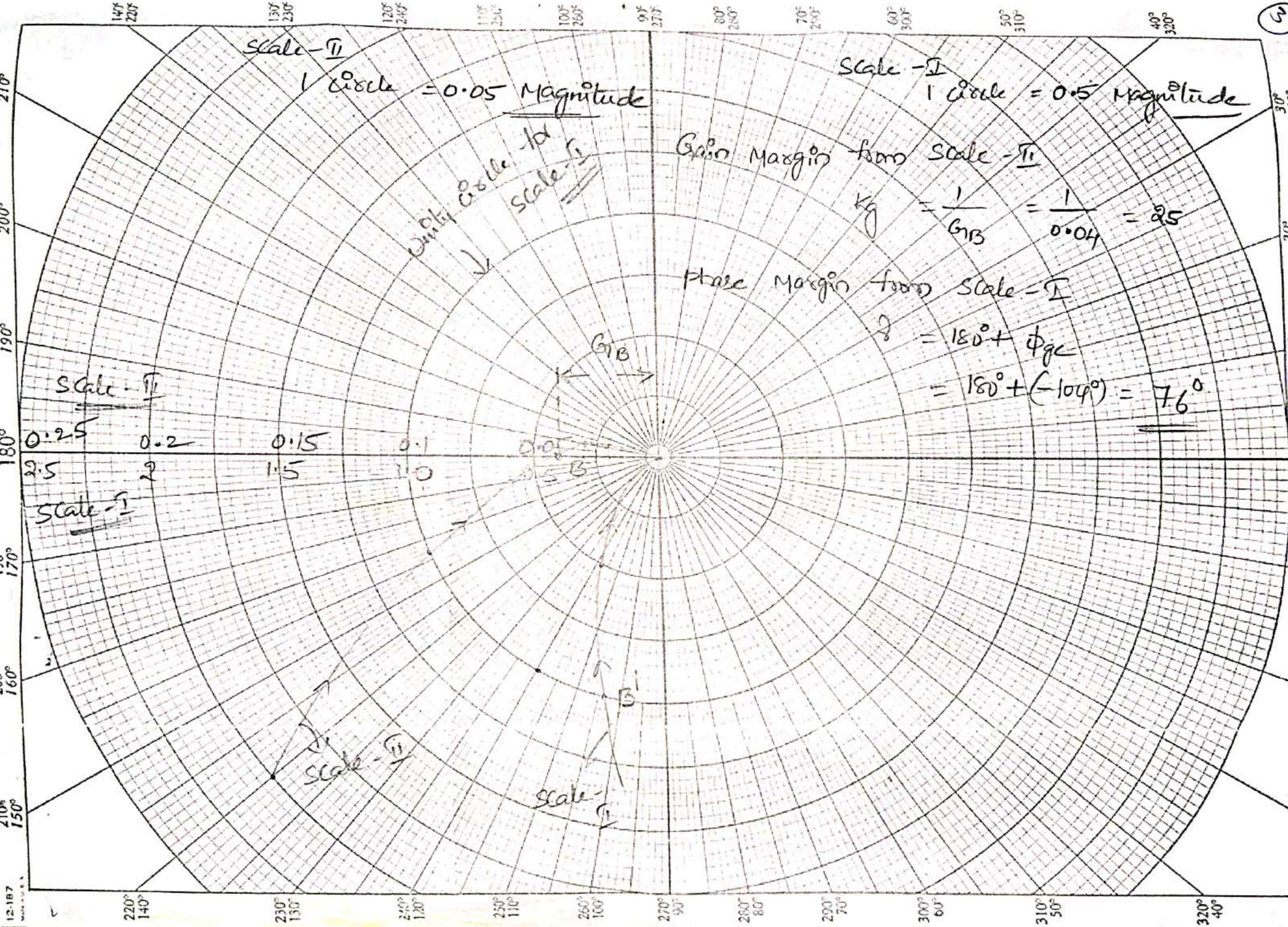
Calculate the magnitude and phase angle from the transfer function $G(j\omega)$.

$$|G(j\omega)| = \frac{1}{\sqrt{\omega^2}\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\text{magnitude } |G(j\omega)| = \frac{1}{\omega\sqrt{1+(0.2\omega)^2}\sqrt{1+(0.05\omega)^2}}$$

$$\angle G(j\omega) = -90^\circ - \tan^{-1}(0.2\omega) - \tan^{-1}(0.05\omega)$$

(3)



Gain Margin from Scale-II

$$K_g = \frac{1}{G_B} = \frac{1}{0.04} = 25$$

Phase Margin from Scale-I

$$\gamma = 180^\circ + \phi_{gc}$$

$$= 180^\circ + (-104^\circ) = \underline{\underline{76^\circ}}$$

Now Calculate the magnitude and phase angle for the various Corner frequencies.

ω	2	1	4	6	8	10	12	14
$\lvert G(j\omega)\rvert$	0.5	1.0	0.2	0.102	0.06	0.04	0.027	0.02
$\underline{/G(j\omega)}$	-117.5	-104	-140	-157	-170°	-180°	-188	-195

By using these polar Co-ordinates plot the graph in the polar graph, by assuming the Proper Scale. And the graph what we obtained is Called polar plot.

From the above polar Co-ordinates convert these into Rectangular Co-ordinates. And tabulate these as shown below,

ω	1	2	4	6	8	10	12	14
$G_R(j\omega)$	-0.24	-0.23	-0.15	-0.092	-0.050	-0.04	-0.026	-0.019
$G_I(j\omega)$	-0.97	-0.44	-0.13	-0.039	-0.01	0	0.003	0.005

Now, Marking all the Real and imaginary parts for the Respective Corner frequencies 'ω' on the ordinary graph then find out the gain margin and phase margin.

From the polar graph

$$\text{Gain Margin } k_g = \frac{1}{G_B}$$ (from scale-II)

$$= \frac{1}{0.04} = 25.$$

$$\text{Phase Margin } \gamma = 180° + \phi_{gc}$$

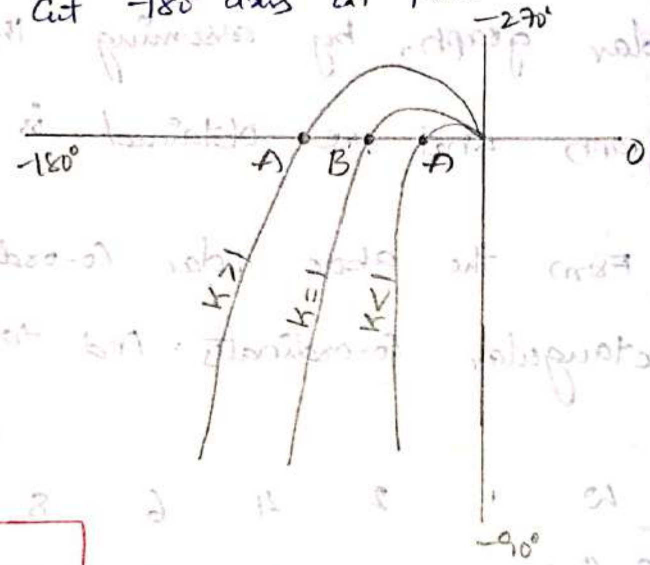
$$= 180° - 104° = 76°.$$ (from scale-I)

But the obtained polar plot is for K=1. Now, obtaining the actual value of K:

To Determine K for specified gain margin :-

Draw $G(j\omega)$ locus with K=1. let it cut the -180° axis at point B corresponding to a gain of G_B.

let the specified gain margin be x dB. For this gain margin, the $G(j\omega)$ locus will cut -180° axis at point A whose magnitude is G_A.



Now, $20\log\frac{1}{G_A} = x$

$\log\frac{1}{G_A} = \frac{x}{20}$

$\frac{1}{G_A} = 10^{x/20}$

$\Rightarrow \boxed{G_A = \frac{1}{10^{x/20}}}$

∴ Now, the value of K is given by, $\boxed{K = \frac{G_A}{G_B}}$

NOTE:- if K<1, then system gain is Reduced.

if K>1, then system gain is increased.

Sol. from the problem, specified gain is 18dB.

$\Rightarrow 20\log\frac{1}{G_A} = 18$

$\Rightarrow \frac{1}{G_A} = 10^{18/20}$

$\Rightarrow G_A = \frac{1}{10^{18/20}} = 0.125$

$\therefore K = \frac{G_A}{G_B} = \frac{0.125}{0.04} = \boxed{3.125 = K}$

* find K when phase Margin is ~~45~~ 60°.

procedure to determine K for specified phase Margin :-

Draw $G(j\omega)$ locus with $K=1$. let it cut the unity circle at point B'. let the specified phase margin be $x°$.

For a phase margin of $x°$, let ϕ_{gc} be the phase angle of $G(j\omega)$ at gain crossover frequency.

$$x° = 180° + \phi_{gc} \qquad \gamma = 180° + \phi_{gc}$$

$$\Rightarrow \phi_{gc} = -180° + x° \qquad 60° = 180° + \phi_{gc} \Rightarrow \phi_{gc} = -120°$$

In the polar plot, the Radial line corresponding to ϕ_{gc} will cut the locus of $G(j\omega)$ with $K=1$ at point A and the magnitude corresponding to the point of G_A.

$$\text{Now,} \quad \boxed{K = \frac{G_B}{G_A} = \frac{1}{G_A}} \qquad [\because G_B = 1]$$

find K for $\gamma =$ ~~45~~ 60°.

Sol. from the above problem

$$\Rightarrow \gamma = 180° + \phi_{gc}$$

$$\Rightarrow \phi_{gc} = ~~4~~60° - 180°$$

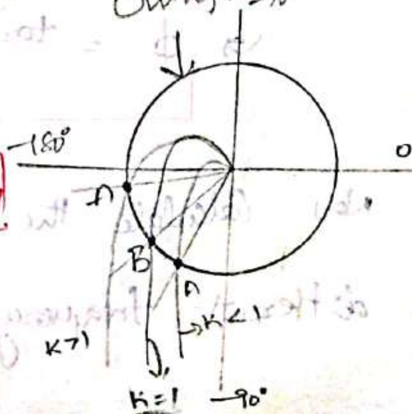
$$= -120°$$

In the polar plot $-120°$ line cuts at the magnitude circle 0.425. let it be $G_A = 0.425$

$$\text{Now,} \quad K = \frac{G_B}{G_A} \qquad [\because G_B = 1]$$

$$= \frac{1}{G_A} = \frac{1}{0.425}$$

$$= \boxed{2.353 = K}$$



4. Sketch the polar plot for the transfer function

$$G(s)\,H(s) = \frac{10\,(s+2)(s+4)}{s\,(s^2-3s+10)}.$$

Sol: given, transfer function $G(s)H(s) = \dfrac{10(s+2)(s+4)}{s\,(s^2-3s+10)}.$

$$G(j\omega)\,H(j\omega) = \frac{10\,(2+j\omega)(4+j\omega)}{j\omega\left[(j\omega)^2-3j\omega+10\right]}$$

$$= \frac{10\,(2+j\omega)(4+j\omega)}{j\omega\,(-\omega^2-3j\omega+10)}$$

$$= \frac{10\,(2+j\omega)(4+j\omega)}{j\omega\left[10-\omega^2+j(-3\omega)\right]}.$$

Now, Magnitude $= |G(j\omega)\,H(j\omega)|$

$$= \frac{10\,\sqrt{2^2+(\omega)^2}\,\sqrt{4^2+\omega^2}}{\sqrt{\omega^2}\,\sqrt{(10-\omega^2)^2+(-3\omega)^2}}$$

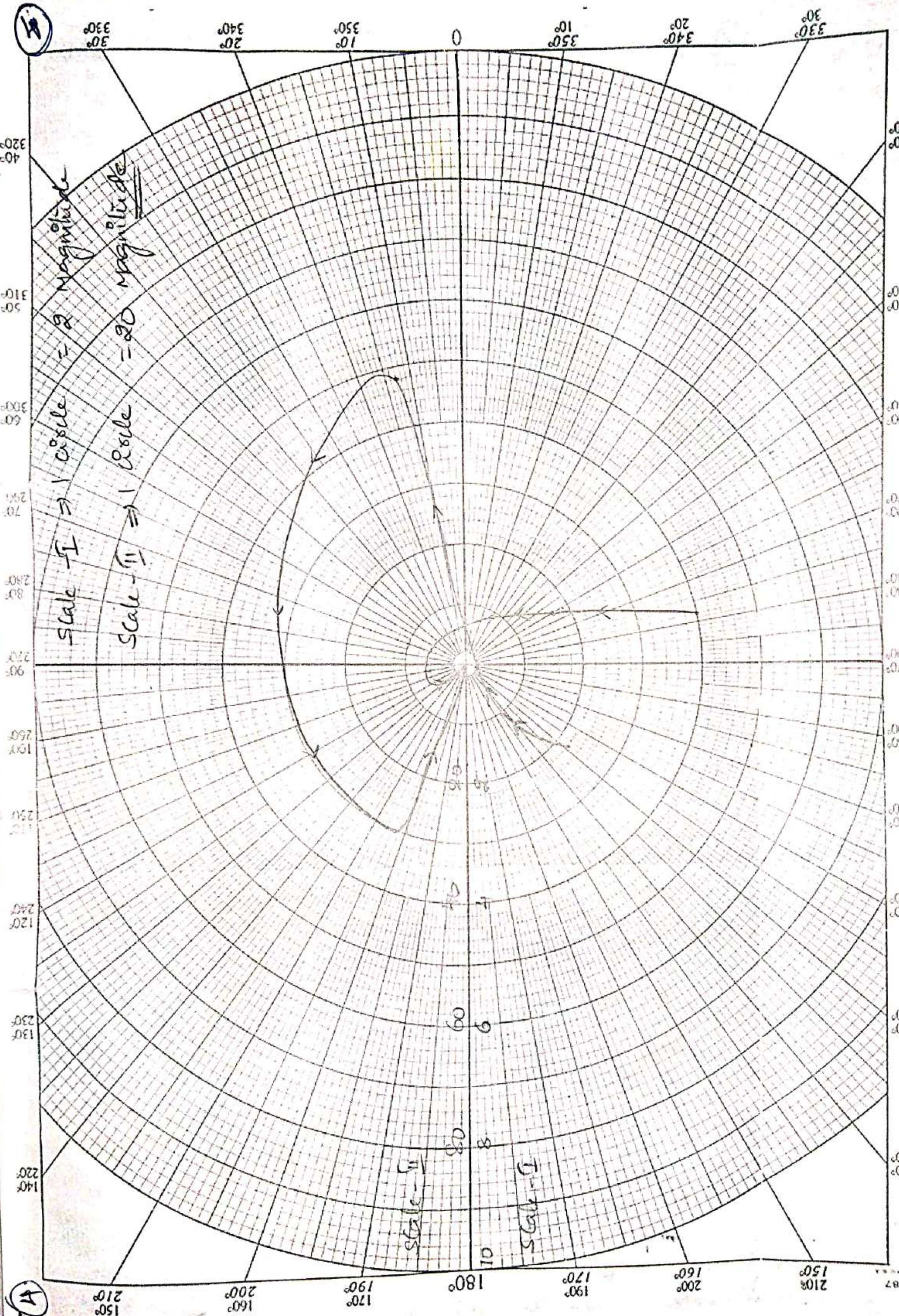
$$\boxed{|G(j\omega)\,H(j\omega)| = \frac{10\,\sqrt{4+\omega^2}\,\sqrt{16+\omega^2}}{\omega\,\sqrt{(10-\omega^2)^2+9\omega^2}}}$$

and, Phase angle $\angle G(j\omega)\,H(j\omega)$

$$= -90^\circ - \tan^{-1}\left(\frac{-3\omega}{10-\omega^2}\right) + \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right)$$

$$\Rightarrow \boxed{\phi = \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right) - 90^\circ - \tan^{-1}\left(\frac{-3\omega}{10-\omega^2}\right).}$$

Now calculate the magnitude and phase angle values for different frequency values. And is tabulated as below.


Scale - I ⇒ 1 circle = 2 Magnitude
Scale - II ⇒ 1 circle = 20 Magnitude
Scale - II
Scale - I
0
180°

The Magnitude & Angle Values in polar Co-ordinates.

ω	0.1	0.2	0.5	1.	4	5	10	50.
$\lvert G(j\omega)\rvert$	80	40.33	16.85	9.72	4.71	3.25	1	0.2
$\angle G(j\omega)$	$-84°$	$-78°$	$-66°$	$-30.96°$	$13.5°$	$164.54°$	$-141.54°$	$-100°$

angle for the quadratic equation. $(s^2-3s+10)$ will be,

$$\Rightarrow (j\omega)^2 - 3j\omega + 10 \Rightarrow 10-\omega^2 - 3j\omega .$$

$$\Rightarrow \phi = \tan^{-1}\left[\frac{-3\omega}{10-\omega^2}\right].$$

for $\omega \le \omega_n$ $\quad \phi = \tan^{-1}\left(\frac{-3\omega}{10-\omega^2}\right)$

for $\omega > \omega_n$ $\quad \phi = \left[\tan^{-1}\left(\frac{-3\omega}{10-\omega^2}\right) + 180°\right].$

$$\boxed{\begin{aligned} &\therefore \text{for } \omega \le \omega_n \Rightarrow \phi = \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right) - 90° - \tan^{-1}\left(\frac{-3\omega}{10-\omega^2}\right). \\ &\text{for } \omega > \omega_n \Rightarrow \phi = \tan^{-1}\left(\frac{\omega}{2}\right) + \tan^{-1}\left(\frac{\omega}{4}\right) - 90° - \left[\tan^{-1}\frac{-3\omega}{10-\omega^2} + 180°\right]. \end{aligned}}$$

Now, for quadratic term $(s^2-3s+10)$.

equating this to General second order System.

$$\Rightarrow s^2 - 3s + 10 = s^2 + 2\omega_n s \cdot \xi + \omega_n^2$$

$$\Rightarrow \omega_n^2 = 10$$

$$\omega_n = \sqrt{10}$$

$$\therefore \boxed{\omega_n = 3.162}$$

$$2\omega_n \xi = -3$$

$$\Rightarrow \xi = -0.474.$$

Then obtain values for 'ω' (the magnitudes & phase angles). Mark on the graph. There by we will get polar plot.

(5) Sketch the polar plot for the transfer function

$G(s) = \frac{200(s+2)}{s(s^2+10s+100)}$. Determine phase Margin and gain Margin ?

Sol given T.F $= \frac{200(s+2)}{s(s^2+10s+100)}$.

Now, Equate the quadratic equation to general second order System $2\xi\omega_n s + s^2 + \omega_n^2 = 0$

$\Rightarrow s^2+10s+100 = 2\xi\omega_n s + \omega_n^2 + s^2$

$\Rightarrow \omega_n^2 = 100$ | $2\xi\omega_n = 10$

$\xi\omega_n = 5$

$\Rightarrow \omega_n = \underline{\underline{10}}$ | $\xi = \frac{5}{10} = 0.5$

Now, Sinusoidal transfer function is,

$$G(j\omega) = \frac{200(j\omega+2)}{j\omega((j\omega)^2+10j\omega+100)}$$

$$= \frac{200\times 2\,(j\omega/2+1)}{j\omega\, 100\left[\frac{s^2}{100}+\frac{10s}{100}+1\right]}$$

$$= \frac{200\times 2\,[0.5j\omega+1]}{100\, j\omega\,[0.01(j\omega)^2+0.1j\omega+1]}$$

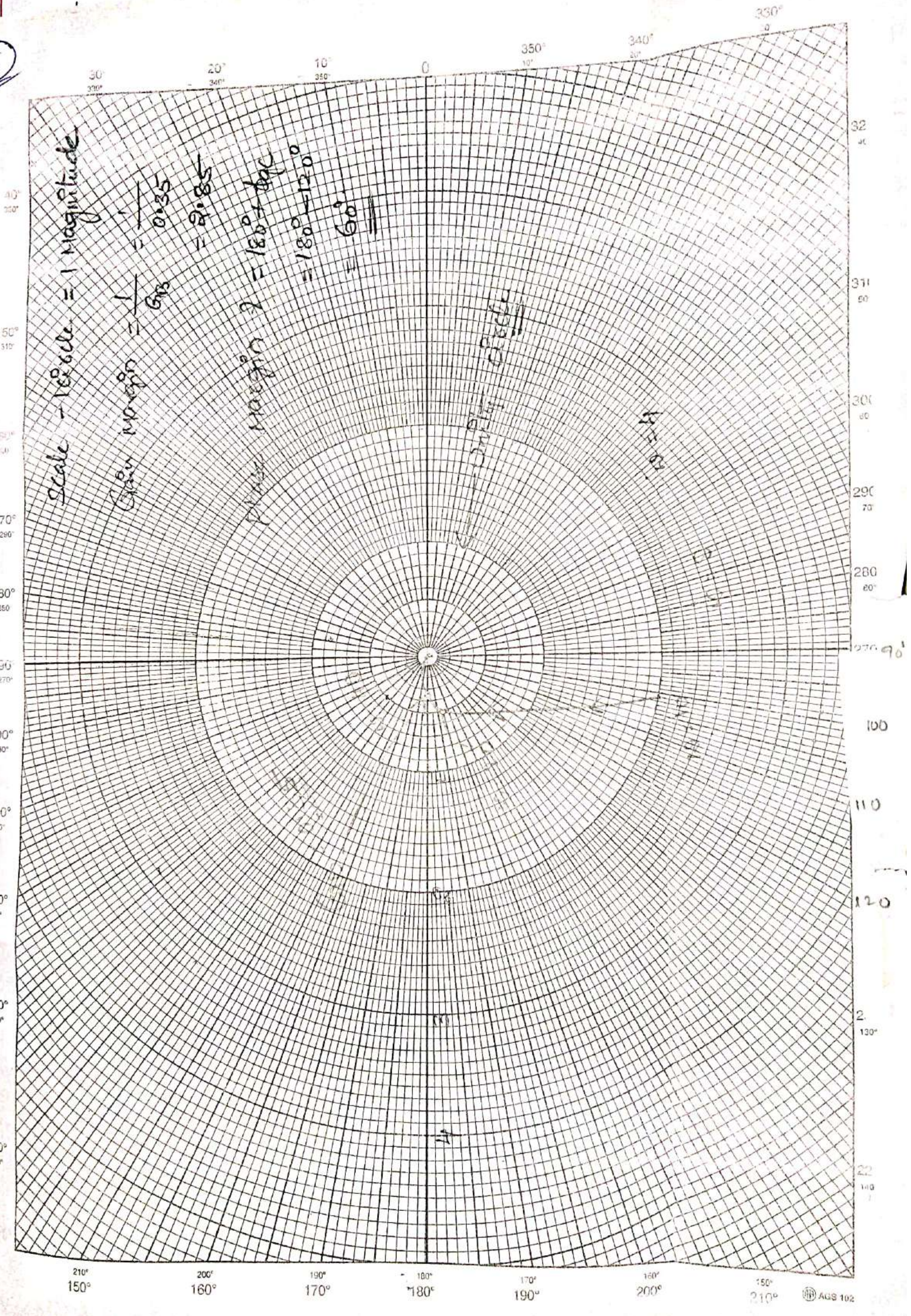
$$= \frac{4\,(1+0.5j\omega)}{j\omega\,(0.1j\omega+1-0.01\omega^2)}$$

Now,

Magnitude $|G(j\omega)| = \frac{4\sqrt{(0.5\omega)^2+1}}{\omega\sqrt{(1-0.01\omega^2)^2+(0.1\omega)^2}}$

Phase angle $\phi = -90^\circ + \tan^{-1}(0.5\omega) - \tan^{-1}\left[\frac{0.1\omega}{1-0.01\omega^2}\right]$.

5



Scale - 1 circle = 1 Magnitude
Gain margin = $\frac{1}{G_B} = \frac{1}{0.35}$
$= 2.85$
Phase margin $\gamma = 180^\circ + \phi_{gc}$
$= 180^\circ - 120^\circ$
$= 60^\circ$
Unit circle
AGS 102

for $\omega \leq \omega_n$

$$\phi = -90^\circ + \tan^{-1}(0.5\omega) - \tan^{-1}\left[\frac{0.1\omega}{1-0.01\omega^2}\right].$$

for $\omega > \omega_n$.

$$\phi = -90^\circ + \tan^{-1}(0.5\omega) - \left[\tan^{-1}\frac{0.1\omega}{1-0.01\omega^2} + 180^\circ\right]$$

ω	4	6	8	10	15	20	40.
$\lvert G(j\omega)\rvert$	2.40		2.349	2.0396	-	0.557	0.128
$\angle G(j\omega)$	-52.02		-80°	-101.30		-152°	-198°

From the above polar co-ordinates Mark the points on the polar graph and connect those points with a free hand.

The plot is called polar plot.

From the polar graph,

$$\text{gain margin } k_g = \frac{1}{G_{1B}}$$

$$= \frac{1}{0.35}$$

$$= 2.85$$

$$\text{phase Margin } \gamma = 180^\circ + \phi_{gc}$$

$$= 180^\circ - 120^\circ$$

$$= \underline{\underline{60^\circ}}.$$

6. Sketch the polar plot for the following transfer function.

$$G(s) = \frac{e^{-0.1s}}{s(s+1)(s+5)}.$$

Sol. given transfer function $G(s) = \dfrac{e^{-0.1s}}{s(s+1)(s+5)}.$

$$G(j\omega) = \frac{e^{-0.1j\omega}}{j\omega(j\omega+1)(5+j\omega)}.$$

$$G(j\omega) = \frac{e^{-0.1j\omega}}{j\omega(1+j\omega)5(1+0.2j\omega)}$$

$$G(j\omega) = \frac{0.2\, e^{-0.1j\omega}}{j\omega(1+j\omega)(1+0.2j\omega)}$$

Now, phase angle and magnitude from the above transfer function is given by,

$$|G(j\omega)| = \frac{0.2}{\omega\sqrt{1+\omega^2}\sqrt{1+(0.2\omega)^2}}$$

$$|G(j\omega)| = \frac{0.2}{\omega\sqrt{1+\omega^2}\sqrt{1+0.04\omega^2}}$$

phase angle $\angle G(j\omega) = -0.1\omega \times \frac{180}{\pi} - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.2\omega)$

$$\Rightarrow \phi = -0.1\omega \times 57.29 - 90^\circ - \tan^{-1}(\omega) - \tan^{-1}(0.2\omega).$$

Now, calculate the magnitude and phase angle values for various values of frequency 'ω'. The table is shown below.

ω	0.2	0.4	0.6	1.0	2	3	5
$\lvert G(j\omega)\rvert$	0.979 $\cong$ 1.0	0.462 $\cong$ 0.5	0.283 $\cong$ 0.3	0.13	0.04	0.01	0.005
$\angle G(j\omega)$	-104.74	-118.66	-131.24	-152.03	-186.69	-209.71	-242.33

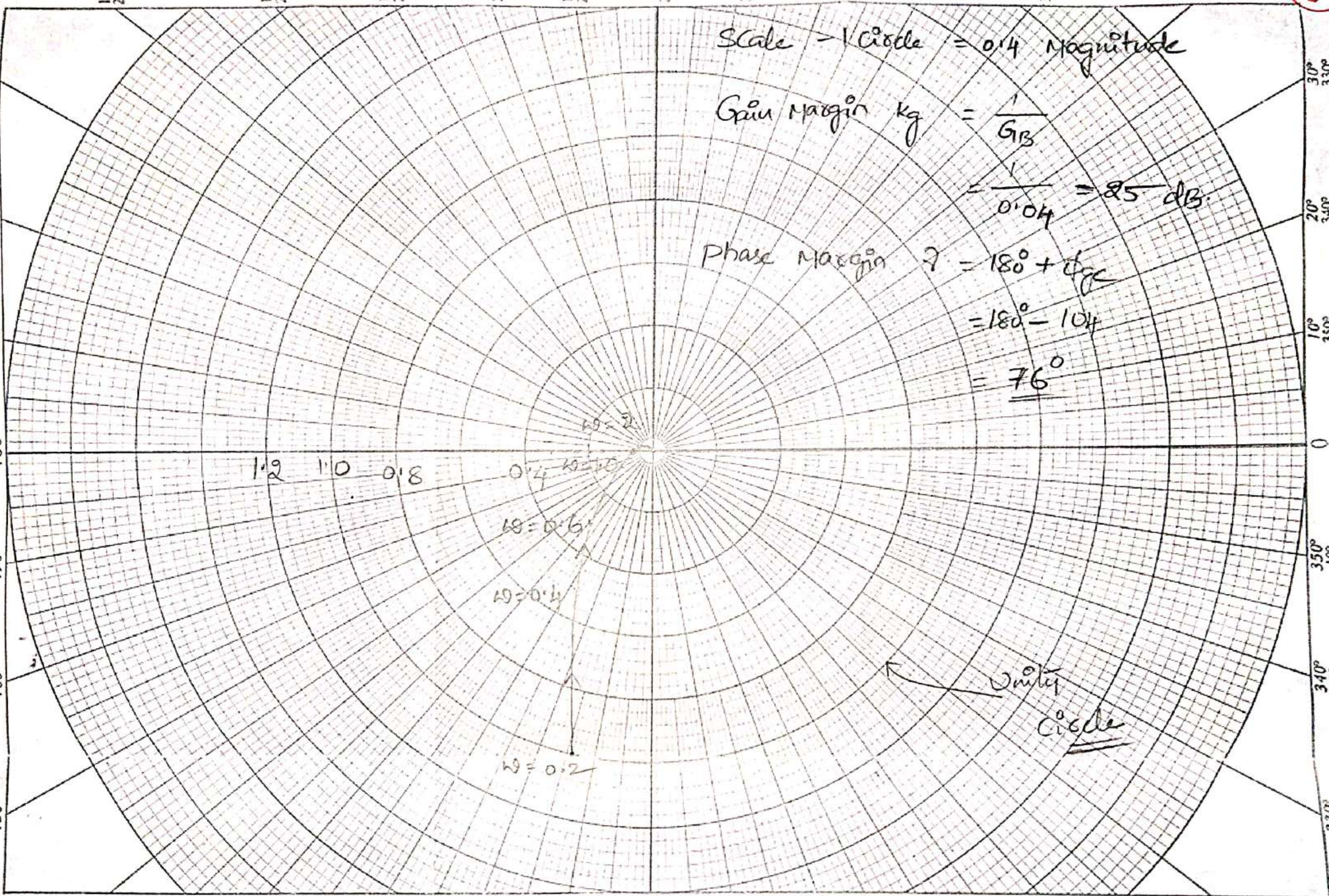
from this values mark the points on the graph, then connect those points with the smooth curve. Then find the phase margin, and gain margin.

$\therefore$ from the graph, gain margin $= K_g = \frac{1}{0.021} = \cancel{50dB}$ 25dB.

phase margin $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ + (-104) = \underline{76^\circ}$

9



Scale - 1 circle = 0.4 Magnitude

Gain Margin $K_g = \dfrac{1}{G_B}$

$= \dfrac{1}{0.04} = 25$ dB.

Phase Margin $\gamma = 180^\circ + \phi_{gc}$

$= 180^\circ - 104$

$= \underline{76^\circ}$

(7) sketch the polar plot for the transfer function $G(S) = \frac{10(S+1)}{(S+10)^2}$, determine phase margin and gain margin.

Sol. given transfer function $G(S) = \frac{10(S+1)}{(S+10)^2}$

$$G(j\omega) = \frac{10(1+j\omega)}{(10+j\omega)^2}$$

$$= \frac{10(1+j\omega)}{(10+j\omega)(10+j\omega)}$$

$$= \frac{10(1+j\omega)}{100(1+0.1j\omega)(1+j\omega 0.1)}$$

$$= \frac{0.1(1+j\omega)}{(1+0.1j\omega)(1+0.1j\omega)}$$

Now find out the magnitude and phase angle for the above transfer function $G(j\omega)$.

$$|G(j\omega)| = \frac{0.1\sqrt{1+\omega^2}}{\sqrt{1+(0.1\omega)^2}\sqrt{1+(0.1\omega)^2}}$$

$$\therefore |G(j\omega)| = \frac{0.1\sqrt{1+\omega^2}}{\sqrt{1+0.01\omega^2}\sqrt{1+0.01\omega^2}} = \frac{0.1\sqrt{1+\omega^2}}{[1+(0.01\omega^2)]}.$$

Now, phase angle $\phi = \tan^{-1}(\omega) - \tan^{-1}(0.1\omega) - \tan^{-1}(0.1\omega)$

$$\therefore \phi = \tan^{-1}(\omega) - 2\tan^{-1}(0.1\omega).$$

Now, take some frequency values (ω) and calculate the magnitudes and phase angles for the respective frequencies and tabulate them which is shown below

The Magnitude and Phase angle in Polar Co-ordinates.

ω	10	20	30	50	100
$\lvert G(j\omega)\rvert$	0.5	0.4	0.3	0.2	0.1
$\angle G(j\omega)$	-5.71	-40°	-55°	-68.5	-80°.

Now, Mark all the points of Magnitudes and Phase angles with respect to frequency values. Then Connect those points with Smooth Curve. And, the plot is known as polar plots.

* Stability Determination of Polar plot :-

As we have seen from the bode plot the stability of a System depends upon the phase margin and gain margin. If both the phase and gain margins are positive, then the System is said to be in stable condition.

Now, In polar plot consider the figure below.

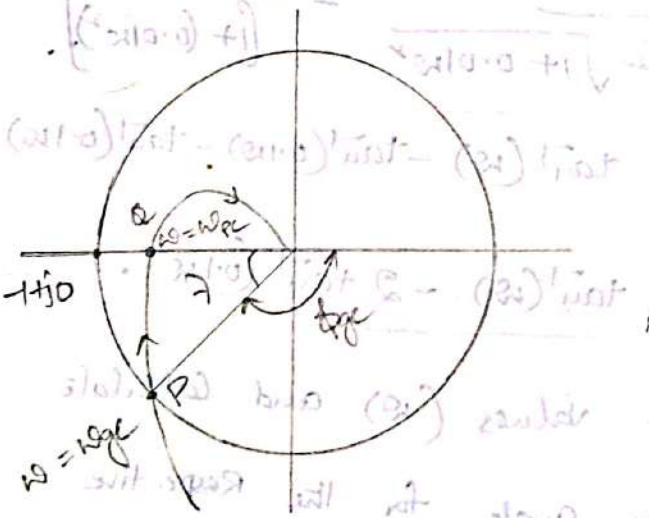


fig (a)

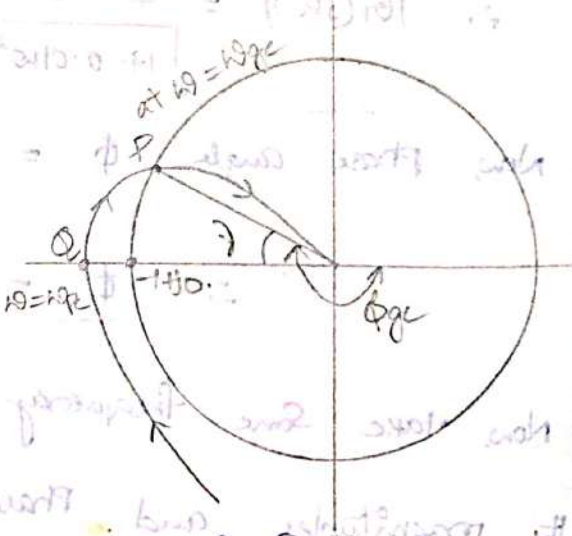
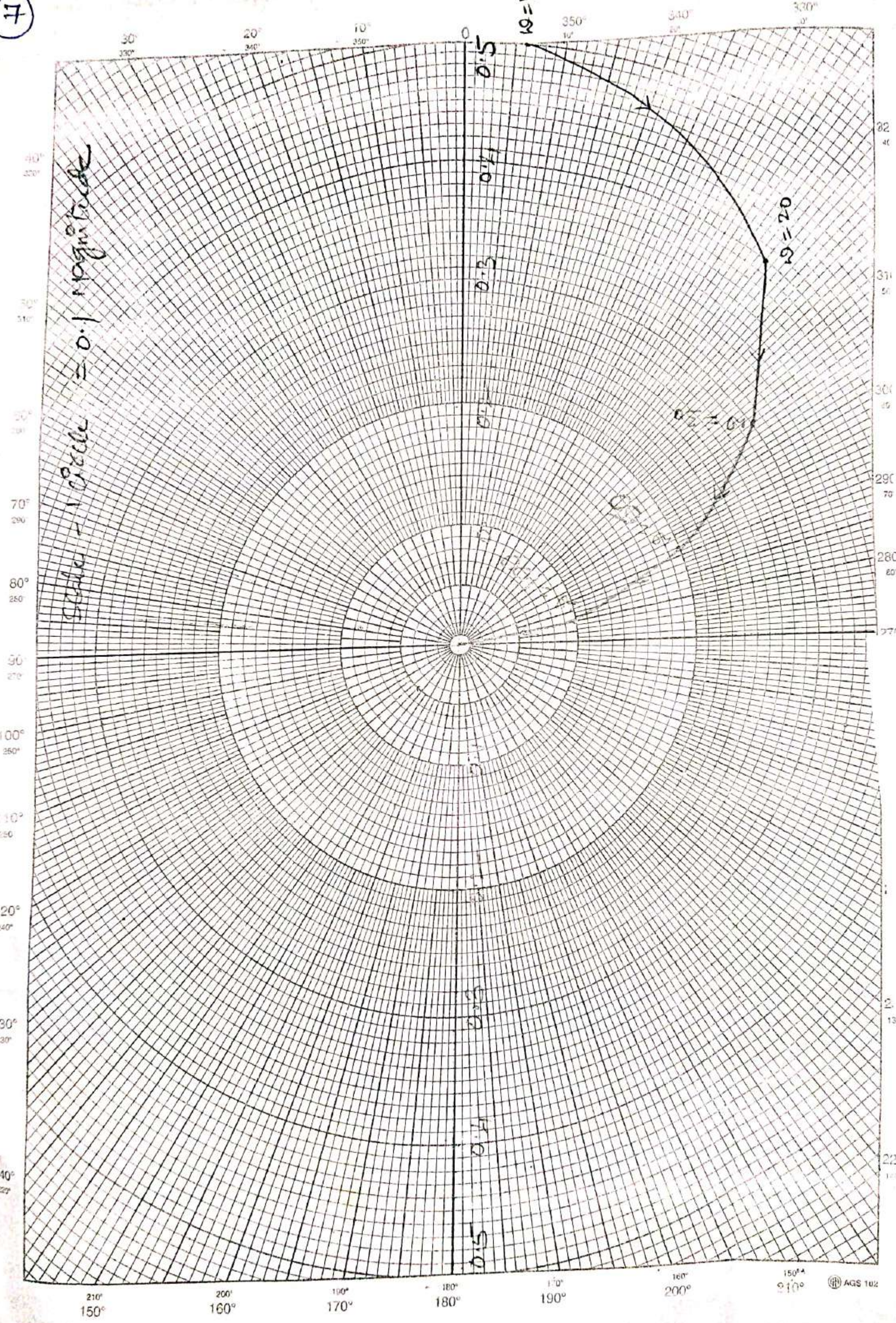


fig. (b)

It is observed that the critical point $-1+j0$ is outside the polar plot intersection to -180° in fig. (a). Such type of

7

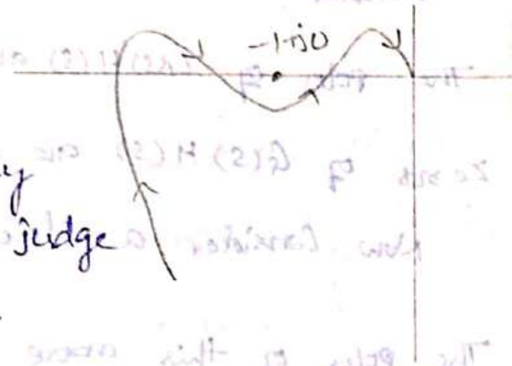


Systems are known as stable systems.

And for un-stable systems the critical point $-1+j0$ is inside the polar plot which is shown in fig. (b). This is known as Enclosement. If critical point is $(-1+j0)$ enclosed by polar plot, system is unstable and if it is not enclosed it is stable in Nature. A portion is said to be enclosed with in the polar plot if it lies to the right hand side when one travels along the polar plot from the point corresponding to $\omega = 0$ to the point corresponding to $\omega = \infty$.

Note :-

Though P.M, G.M are simple concepts to be applied to determine stability, in case of polar plots as shown below in the fig. It is very difficult to determine G.M, P.M and stability.



Such systems are conditionally stable and it is difficult to judge its stability from the polar plot.

The Nyquist plot plays an important role in stability analysis of such systems.

The Nyquist plot provides more clear answer about the stability hence in practice Nyquist criterion is used for stability analysis and not the polar plot.

* Nyquist Plot :-

The Concept of Nyquist Plot is based on the polar plot which can be conviniently applied to the stability analysis of any kind of system. To understand the Nyquist analysis the concept is divided into following sections

1. pole - zero configuration of Nyquist plot point of view.
2. Concept of encirclement
3. Analytic function and its singularities.
4. Mapping Theorem (or) principle of Argument.
5. Nyquist stability criterion.

Pole - zero configuration :-

Any function which can be expressed as a ratio of two polynomials has its own poles and zeros.

Consider function $G(s)H(s)$ called open loop T.F of a system. The poles of $G(s)H(s)$ are known as open loop poles and similarly zeros of $G(s)H(s)$ are called as open loop zeros.

Now Consider a closed loop T.F $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$.

The poles of this above T.F are the roots of C.E $1+G(s)H(s)=0$ are called closed loop poles of a system.

for eg: $G(s)H(s) = \frac{10}{s(s+4)}$

Now, open loop poles are 0, −4 & zeros are absent

while ⇒ $\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$

$$1+G(s)H(s) = 1+\frac{10}{s(s+4)} = \frac{s^2+4s+10}{s(s+4)}.$$

⇒ closed loop Transfer function $\dfrac{C(S)}{R(S)} = \dfrac{\dfrac{\quad}{S(S+4)}}{\dfrac{S^2+4S+10}{S(S+4)}}$

Now, closed loop poles are the roots of equation $S^2+4S+10=0$.

from $1+G(S)H(S) = \dfrac{S^2+4S+10}{S(S+4)} = \dfrac{P(S)}{Q(S)}$.

now, roots of $P(S)=0$ are zeros of $1+G(S)H(S)$. Similarly, $Q(S)=0$ are the roots which are called poles of $1+G(S)H(S)$, which is similar to ~~closed~~ open loop poles.

∴ poles of $1+G(S)H(S)$ = open loop poles of a system

Zeros of $1+G(S)H(S)$ = closed loop poles of a system

Encirclement :-

It is applicable for the paths which are open and not closed like polar plots (enclosement). But for closed path, it is necessary to understand the concept of an encirclement.

A point is said to be encircled by a closed path if it is found to lie inside that closed path. fig.ⓐ

This is a simple concept but in some complicated closed paths it is possible that point lies inside the path but actually not encircled by the path. Hence it is always better to count the number of encirclements of a point.

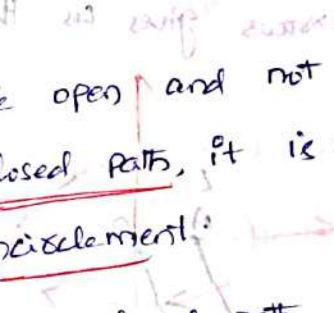


fig.ⓐ

Counting Number of Encirclements :

In the fig.ⓐ it can be easily concluded that the number of encirclements of point A and O, is one in anti-clockwise direction. But for complicated cases, it is not possible to judge number of encirclements by mere inspection.

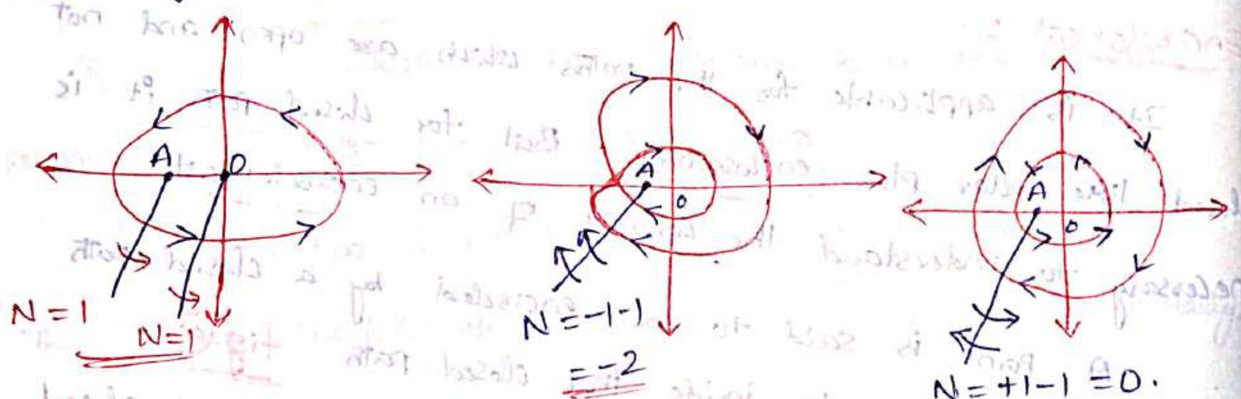
Method to judge the number of Encirclements :-

1. Draw a vector from a point whose encirclements are to be determined, in such a way to join any point outside that closed path in any direction. Avoid confusing Directions.

2. Identify the number of intersections of this vector with a closed path.

3. mark these intersections with small arrow on same vector indicating direction of closed path at the time of intersection.

4. Cancel the oppositely directed encirclements. The Remaining arrows gives us the number of encirclements of that point.



Note :-

Anticlockwise encirclements are treated as positive.

Clockwise encirclements are treated as Negative

Analytic function and Singularities :-

A Mathematical function is said to be analytic at a point in a plane if its value and its derivative has finite value at that point.

If a point in a plane, the value of function or its derivative is infinite, the function is said to be non-analytic at that point and such a point is called singularity of the function.

Consider a function $F(s) = \frac{25}{s(s+2)}$. Then it is analytic at all points in s-plane except the point $s=0$ and $s=-2$. This is because the function $F(s)$ has a value infinity at $s=0, s=-2$ which are the poles of function $F(s)$ it self.

In general the poles of the function are its singularities.

Similarly, one more function we will define is single valued function.

A function $F(s)$ who has one and only one value for each seperate value of s is said to be single valued.

Consider, $F(s) = \sqrt{s}$ for $s=25$, $F(s)$ has two values $+5$ and -5.

Such a function is not single valued.

In our Control Systems analysis we will assume that the transfer function of the system (or) functions $G(s)$ and $H(s)$ are single valued.

Mapping Theorem (or) principle of Argument :-

The Mapping Theorem States that, let $F(s)$ a single valued function, analytic at all points in s-plane except some finite number of points. These are singularities of function $F(s)$, where it is not analytic. Consider an arbitrary path $T(s)$ in s-plane in such a way that the function $F(s)$ is analytic at each and every point on $T(s)$ path. So the restriction to select a closed path is that it should not pass through the points which are singularities of $F(s)$. i.e. it should not pass through the poles of $F(s)$.

Now, let P and Z be the number of Poles and zeros of $F(S)$ which are encircled by $T(S)$ path i.e. which are inside path $T(S)$. we are not interested in all the poles and zeros of but only those which are encircled by $T(S)$ path in s-plane.

Now, P = Number of Poles of $F(S)$ encircled by $T(S)$.

Z = Number of Zeros of $F(S)$ encircled by $T(S)$.

Now, according to the Mapping procedure, the closed path T in s-plane can be mapped into other plane say - F-plane & get the closed path say $T'(S)$.

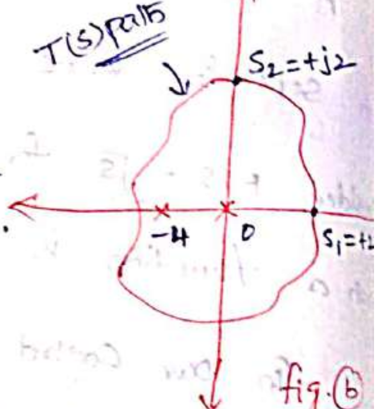


fig. 6

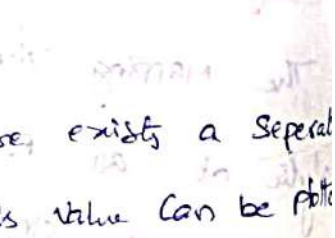
Mapping procedure :-

Consider a function $F(S) = \dfrac{10}{S(S+4)}$.

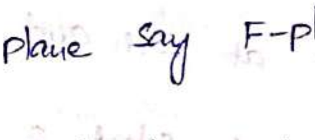
Now we have selected a closed path in s-plane in such a way to encircle both the poles at $S=0$ and $S=-4$ shown in fig. 6

The function $F(S)$ is analytic at all the points on $T(S)$ path selected. And $P=2$ as both the poles are encircled by $T(S)$ path selected.

Now, for each point on $T(S)$ path, there exists a seperate value of $F(S)$ as it is single valued. This value can be plotted in other plane say F-plane.



Eg: at $S_1 = +2 \Rightarrow F(S) = \dfrac{10}{2(2+4)}$

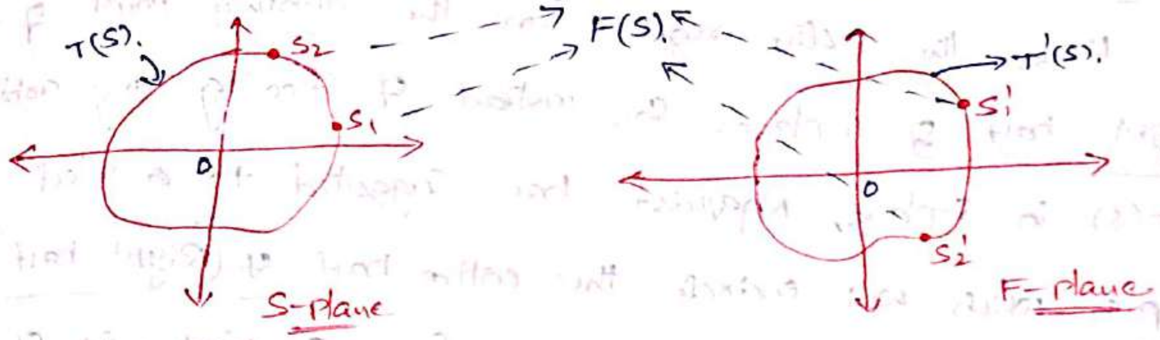


$= \dfrac{10}{12} = 0.834.$

at $S_2 = +j2 \Rightarrow F(S) = \dfrac{10}{j2(4+j2)} = -0.5 - 1j.$

similarly, all points on T(S) path can be plotted in F-plane by finding the value of F(S) at all these points.

This procedure is called mapping of a point from one plane to other plane and F(S) is called mapping function.



Mapping Theorem statement :-

Mapping theorem states that the mapped locus T'(S) encircles the new origin of F-plane as many times as the difference between the number of zeros and poles of F(S) which are encircled by T(S) path in S-plane.

Mathematically, $N = Z - P$.

N = Number of encirclements of origin of F-plane by T'(S) path.

P = Number of poles of F(S) encircled by T(S) path in S-plane

Z = Number of zeros of F(S) encircled by T(S) path in S-plane

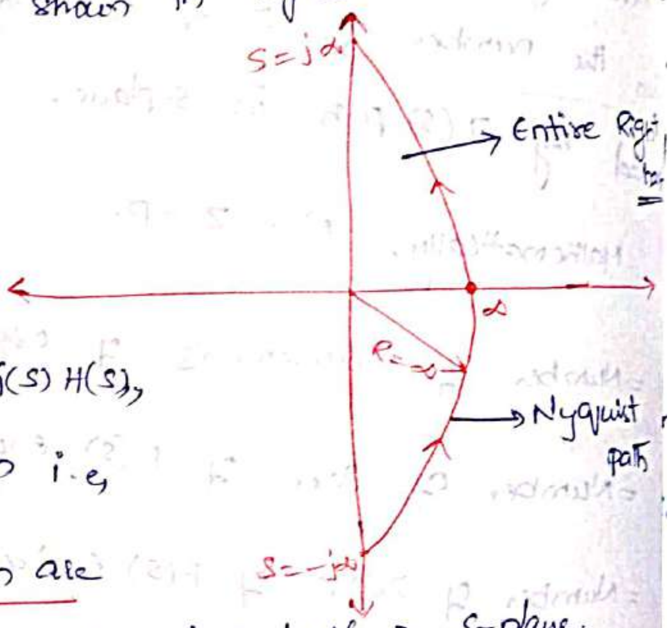
This statement is also called as principle of Argument

Nyquist stability criterion :-

For stability, all the zeros of $1+G(S)H(S)$ must be in the left half of S-plane, none of the zeros should be in the right half of S-plane.

Nyquist has suggested that rather than analyzing whether all the zeros are located in the left half of S-plane, it is better to examine the presence of any one zero of $1+G(s)H(s)$ in right half of S-plane making system unstable.

Hence, the active region from the stability point of view is right half of S-plane. So, instead of choosing any arbitrary path $T(s)$ in S-plane, Nyquist has suggested to select a $T(s)$ path which will encircle the entire half of (Right half) S-plane. Such a path should start from $s=+j\infty$. It should be continued till $s=-j\infty$ along imaginary axis and then be completed with a semicircle of radius ∞, encircling the right half of S-plane, shown in fig. ⓒ. This path is called as Nyquist path.



Now as poles of $G(s)H(s)$ are known which are the poles of $1+G(s)H(s)$, we know the value of P i.e. poles of $1+G(s)H(s)$ which are encircled by Nyquist path in right half of S-plane.

Now map all the points on the Nyquist path into F-plane with the help of mapping function $1+G(s)H(s)$ to $T'(s)$ plane.

This mapped locus obtained in F-plane by mapping all the points on Nyquist path is called Nyquist plot.

As this locus is obtained, we can determine the number of encirclements of origin by Nyquist plot in F-plane. Say, N.

From the Mapping Theorem these encirclements must satisfy the equation. $N = Z - P$.

Z = number of zeros lies on Right half of s-plane

P = number of poles lies on Right half of s-plane

N = Number of encirclements.

* Nyquist Stability Criterion states that for absolute stability of the System, the number of encirclements of new origin of F-plane by Nyquist plot must be equal to the difference in number of zeros and number of poles lies in Right half of s-plane **

NOTE:-

For ease of Mapping Nyquist path from s-plane to F-plane instead of Considering Mapping function as $1+G(s)H(s)$, it is Considered as $G(s)H(s)$ only.

But due to this, stability criterion remains the same,

i.e, $N = Z - P$.

But the difference is,

N = Number of encirclements of a critical point $-1+j0$ of F-plane by Nyquist plot instead of number of encirclements of an origin.

కానీ Nyquist path $1+G(S)H(S)$ zeros నుండి develop చేయబడి ఇక్కడ మనం $G(S)H(S)$ ని తీసుకోబోతున్నాం, కానీ Direct గా the loop transfer function తీసుకోబోతున్నాం కాబట్టి, number of encirclements 'N' counted for the critical point '$-1+j0$' but not for origin.

*** Step by step procedure to solve Nyquist criterion :-**

1. Count how many number of poles of $G(S)H(S)$ are in Right half of S-plane i.e, with positive real part. This is value of P.
2. Decide the stability criterion as $N = Z - P$ i.e, how many times Nyquist plot should encircle '$-1+j0$' point for the absolute stability.
3. Select Nyquist path as per the function $G(S)H(S)$.
4. Analyse the sections as starting point and terminating point of plot.
5. Mathematically find out ω_{pc} and intersection of Nyquist plot with negative real axis by Rationalizing $G(S)H(S)$.
6. With the information from step 4 and step 5, sketch the Nyquist plot
7. Count the number of Encirclements N of $-1+j0$ by Nyquist plot. If this matches with the criterion decided in step 2, System is stable, Otherwise unstable

Unit - 8

28/8/12

State Space Analysis of Continuous Systems.

* State space Analysis Approach is a powerfull technique for the analysis and design of Control Systems. The state space Analysis is a Modern approach and also easier for analysis using Computers. The Conventional Methods of analysis employs the transfer function of the System.

The Drawbacks of the Transfer function Model are

1. Transfer function is applicable to linear time invariant Systems.
2. Transfer function analysis is restricted to single input and single output systems.
3. Does not provide the information Regarding the internal state of the System

→ The state space Analysis can be applied for any type of the Systems. The analysis can be carried with initial conditions and can be carried on multiple input and multiple output Systems.

Definitions :-

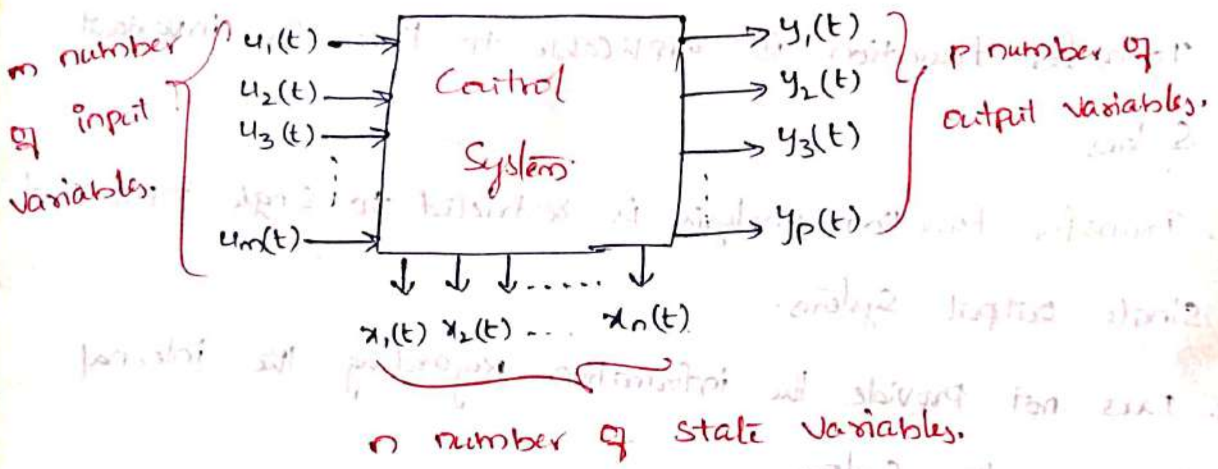
State : The state is the Condition of the System at any instant of time 't'.

State variable :- A set of variable which Describes the state of the System at any time instant is called state variable

* State space Formulation :-

The state of a dynamic system is a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of the inputs for $t \geq t_0$, completely determines the behaviour of the system for $t > t_0$.

In the state variable formulation of a system, in general a system consists of m-inputs, p-outputs and n-state variables.



Input vector $U(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_m(t) \end{bmatrix}$, output Vector $Y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}$

State variable Vector $X(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

* State Model of a linear System :-

The state Model of a System Consists of the state Equation and the output equation. The State equation of a System is a function of state variables and inputs as defined below.

The state equation can be arranged in the form of 'n' number of first order differential equations as below,

$$\frac{dx_1}{dt} = \dot{x}_1 = f_1(x_1, x_2, \dots x_n ; u_1, u_2, \dots u_m)$$

$$\frac{dx_2}{dt} = \dot{x}_2 = f_2(x_1, x_2 \dots x_n ; u_1, u_2 \dots u_m)$$

$$\vdots$$

$$\frac{dx_n}{dt} = \dot{x}_n = f_n(x_1, x_2 \dots x_n ; u_1, u_2, \dots u_m)$$

The above equations can be written as,

$$\boxed{\dot{x}(t) = f[x(t), u(t)]}.$$

For linear time invariant systems the first derivatives of the state variables can be expressed as a linear combination of state variables and inputs.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m$$

When above equations are written in Matrix form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1m} \\ b_{21} & b_{22} & \dots & b_{2m} \\ \vdots & & & \vdots \\ b_{n1} & b_{n2} & \dots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad \text{(A)}$$

In general the above equation can be expressed as,

$$\dot{x} = Ax + BU \quad \text{(or)} \quad \boxed{\dot{x}(t) = Ax(t) + BU(t)}$$

where, $x(t)$ = State Vector Matrix

$U(t)$ = Input vector Matrix.

A = System Matrix

B = Input matrix.

The output at any time are functions of state variables and inputs

$\therefore$ output vector $Y(t) = f[x(t), U(t)]$

i.e, $y_1 = c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n + d_{11}u_1 + d_{12}u_2 + \dots + d_{1m}u_m$

$y_2 = c_{21}x_1 + c_{22}x_2 + \dots + c_{2n}x_n + d_{21}u_1 + d_{22}u_2 + \dots + d_{2m}u_m$

$\vdots$

$y_p = c_{p1}x_1 + c_{p2}x_2 + \dots + c_{pn}x_n + d_{p1}u_1 + d_{p2}u_2 + \dots + d_{pm}u_m$

Now, writing in Matrix form,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & & & \\ c_{p1} & c_{p2} & \dots & c_{pn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1m} \\ d_{21} & d_{22} & \dots & d_{2m} \\ \vdots & & & \\ d_{p1} & d_{p2} & \dots & d_{pm} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

$$\Rightarrow Y(t) = C\,X(t) + D\,U(t). \quad \text{——} (B)$$

Here,

$Y(t)$ = Output Vector Matrix

C = output matrix.

from the equations (A) and (B)

(A) $\Rightarrow \dot{X} = A\,X(t) + B\,U(t)$ $\longrightarrow$ State Equation

(B) $\Rightarrow Y = C\,X(t) + D\,U(t)$. $\longrightarrow$ output Equation

} State Model

* State Diagram :-

The Pictorial Representation of a state Model of the system is called State Diagram. The state Diagram of the System can be either in Block Diagram form (or) signal flow graph form.

The State diagram of a State model is Constructed using three basic elements Scalar, Adder and Integrator

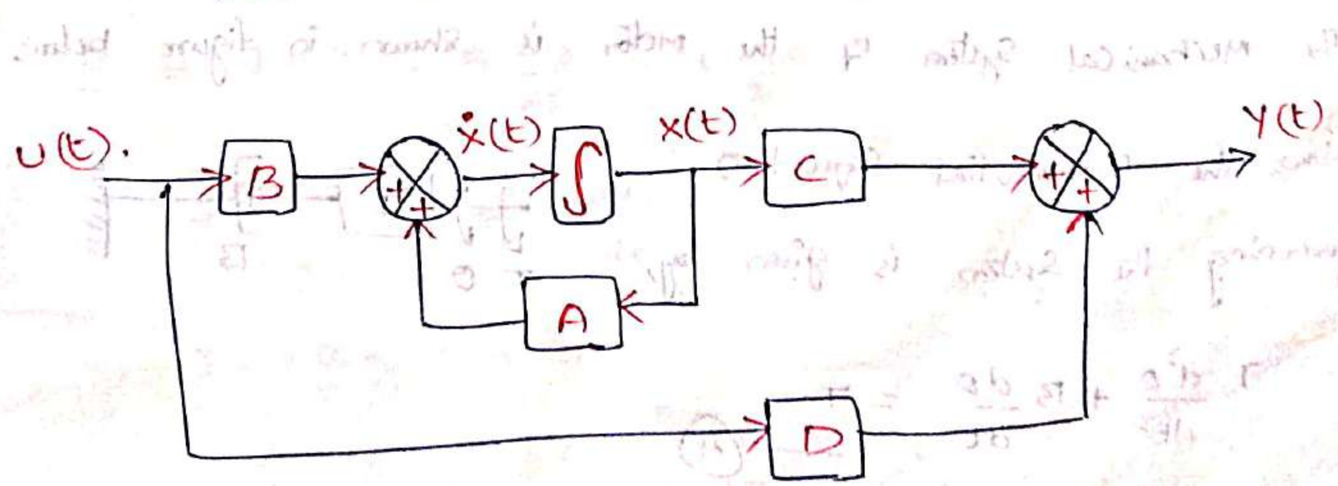
Scalar :- The scalar is used to multiply a signal by a constant. The input signal $x(t)$ is Multiplied by a scaler a to give the output, $a\,x(t)$.

Adder :- The adder is used to add two or more signals. The output of the adder is sum of incoming signals.

Integrator :- The integrator is used to integrate the signal. They are used to integrate the derivaties of state variables to get the state variables.

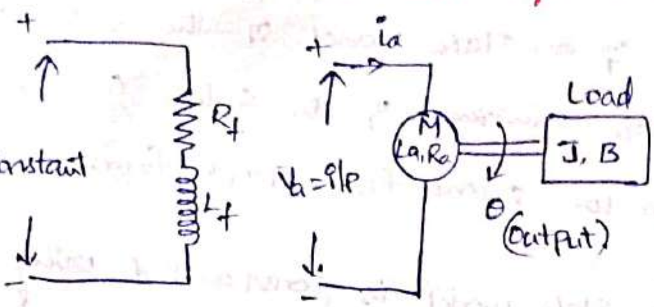
$$\dot{x}(t) = A\,x(t) + B\,u(t)$$

$$y(t) = C\,x(t) + D\,u(t).$$

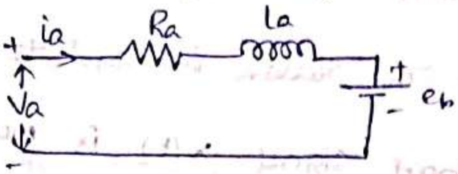


* Determine the state model of Armature Controlled DC Motor.

Sol.



Now, the equivalent circuit of Armature is given by,



By Kirchoff's voltage law,

$$\Rightarrow i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \quad \text{--- (1)}$$

The back emf of DC Machine is proportional to speed of shaft

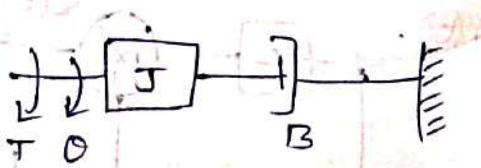
$$\therefore e_b \propto \frac{d\theta}{dt} \Rightarrow e_b = K_b \frac{d\theta}{dt} \quad \text{--- (2)}$$

and Torque of DC Machine is proportional to flux and current. Since flux is constant in this system, the torque is proportional to i_a alone

$$\Rightarrow T \propto i_a \Rightarrow T = K_t i_a \quad \text{--- (3)}$$

The Mechanical system of the motor is shown in figure below,

Now, the Differential Equation governing the system is given by,



$$J \frac{d^2\theta}{dt^2} + B \frac{d\theta}{dt} = T. \quad \text{--- (4)}$$

Now, Substitute (2) in (1)

$$\Rightarrow i_a R_a + L_a \frac{di_a}{dt} + e_b = V_a \Rightarrow i_a R_a + L_a \frac{di_a}{dt} + K_b \cdot \frac{d\theta}{dt} = V_a \quad \text{--- (5)}$$

Similarly Substitute (3) in (4)

$$\Rightarrow J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_t \cdot i_a. \quad \text{—— (6)}$$

Now the equations (5) and (6) are the Differential equations governing the Armature controlled DC Motor.

Now, let choose, i_a, ω, and θ as state variables to model the armature controlled DC Motor.

$$x_1 = i_a, \quad x_2 = \omega = \frac{d\theta}{dt}, \quad x_3 = \theta.$$

The input to the motor is armature voltage V_a and let $V_a = u$, where u is the general notation for input variable

on Substituting the state variables in (5)

$$\Rightarrow (5) \quad = x_1 R_a + L_a\frac{dx_1}{dt} + K_b x_2 = u.$$

let, $\frac{dx_1}{dt} = \dot{x}_1 \quad \Rightarrow \quad x_1 R_a + L_a \cdot \dot{x}_1 + K_b x_2 = u$

$$\Rightarrow \dot{x}_1 = \frac{-R_a}{L_a}x_1 - \frac{K_b}{L_a}x_2 + \frac{1}{L_a}u. \quad \text{—— (7)}$$

Similarly,

$$(6) \Rightarrow J\frac{d^2x_3}{dt^2} + B\frac{dx_3}{dt} = K_t \cdot x_1$$

let $\frac{d^2x_3}{dt^2} = \dot{x}_2$ and $\frac{dx_3}{dt} = x_2$

$$\therefore J\dot{x}_2 + Bx_2 = K_t \cdot x_1.$$

$$\Rightarrow \dot{x}_2 = \frac{K_t}{J}x_1 - \frac{B}{J}x_2 \quad \text{—— (8)}$$

we have assumed $x_3 = \theta$

$$\Rightarrow \frac{dx_3}{dt} = \dot{x}_3 \quad \text{and} \quad \frac{d\theta}{dt} = x_2$$

$$\Rightarrow \dot{x}_3 = x_2 \quad \text{—— (9)}$$

from the ⑦ ⑧, ⑨ equations,

$$\dot{x}_1 = -\frac{R_a}{L_a} x_1 - \frac{K_b}{L_a} x_2 + \frac{1}{L_a} u.$$

$$\dot{x}_2 = \frac{K_t}{J} x_1 - \frac{B}{J} x_2$$

$$\dot{x}_3 = x_2$$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -\frac{R_a}{L_a} & -\frac{K_b}{L_a} & 0 \\ \frac{K_t}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} \\ 0 \\ 0 \end{bmatrix} u.$$

$$\dot{X} = A X + B U.$$

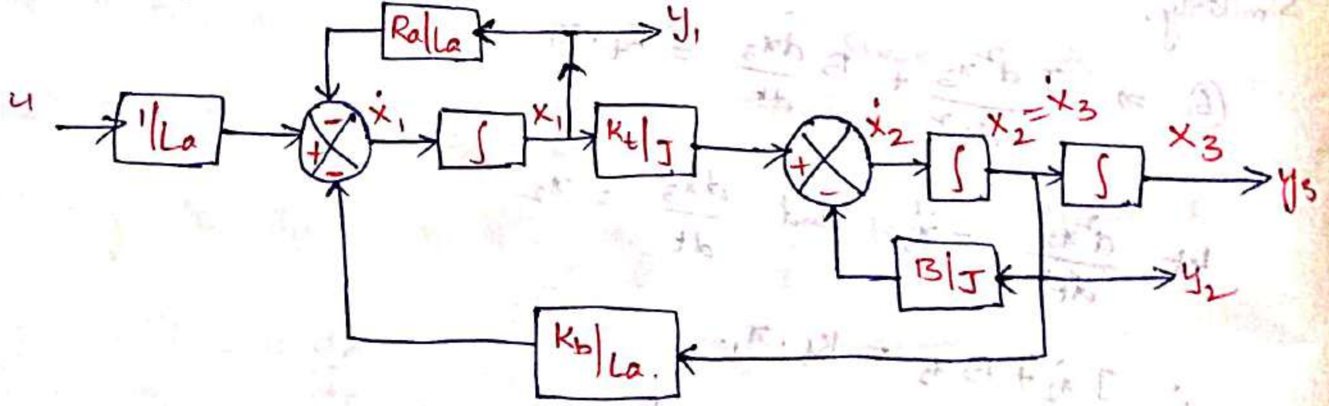
The desired outputs will be i_a, ω and θ.

$$\therefore y_1 = i_a, \quad y_2 = \omega = \frac{d\theta}{dt}, \quad y_3 = \theta$$

$$\Rightarrow y_1 = x_1, \quad y_2 = x_2, \quad y_3 = x_3.$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow Y = CX.$$

Block Diagram Representation



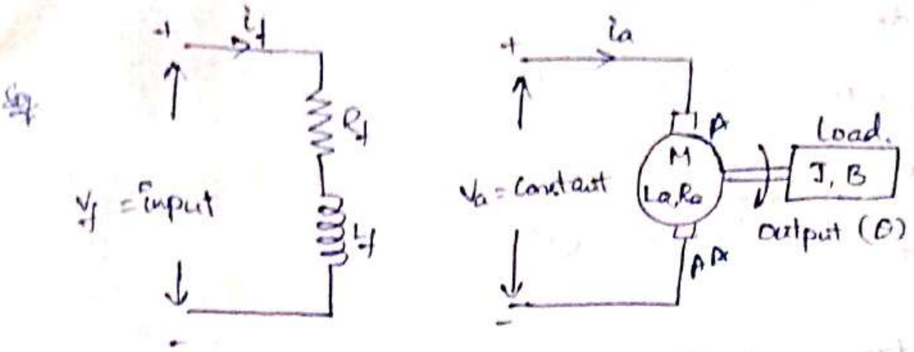
state equation $\dot{X} = AX + BU$

output equation $Y = CX + DU$

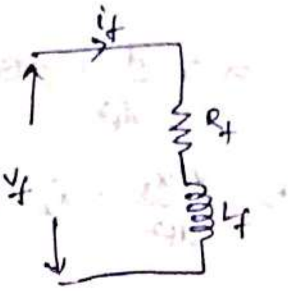
} Combinely called as STATE MODEL

* Determine the state model for field controlled DC motor.

Sol.



Now, the equivalent circuit of field is,



By Kirchoff's Voltage law.

$$R_f i_f + L_f \frac{di_f}{dt} = V_f \quad \text{——} \ (1)$$

The Torque of DC motor is directly proportional to flux and armature current. Since armature current is constant, but flux is directly proportional to field current. So, the torque is proportional to field current

$$\therefore T \propto i_f \Rightarrow T = K_{tf}\, i_f \quad \text{——} \ (2)$$

from the Mechanical system

$$J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = T \quad \text{——} \ (3)$$

Substitute (2) in (3)

$$\Rightarrow J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} = K_{tf}\, i_f \quad \text{——} \ (4)$$

Now, (1) and (4) are the Differential equations governing the system.

let us choose, i_f, ω, θ are the state variables and are Related to the general notation of state variables x_1, x_2 and x_3.

$$\therefore x_1 = i_f, \quad x_2 = \omega = \frac{d\theta}{dt}, \quad x_3 = \theta \ \text{and} \ V_f = u \ (\text{input})$$

on substituting these values in (1) and (4) Equations.

$$① \Rightarrow R_f x_1 + L_f \frac{dx_1}{dt} = u$$

$$\Rightarrow R_f x_1 + L_f \dot{x}_1 = u$$

$$\Rightarrow \dot{x}_1 = \frac{-R_f}{L_f} x_1 + \frac{1}{L_f} u. \quad — ⑤$$

$$④ \Rightarrow J \frac{d^2 x_3}{dt^2} + B \frac{dx_3}{dt} = K_{tf} x_1$$

let, $\frac{d^2 x_3}{dt^2} = \dot{x}_2$ and $\frac{dx_3}{dt} = x_2$

$$\Rightarrow J \dot{x}_2 + B x_2 = K_{tf} x_1.$$

$$\Rightarrow \dot{x}_2 = \frac{K_{tf}}{J} x_1 - \frac{B}{J} x_2 \quad — ⑥$$

$x_3 = \theta$

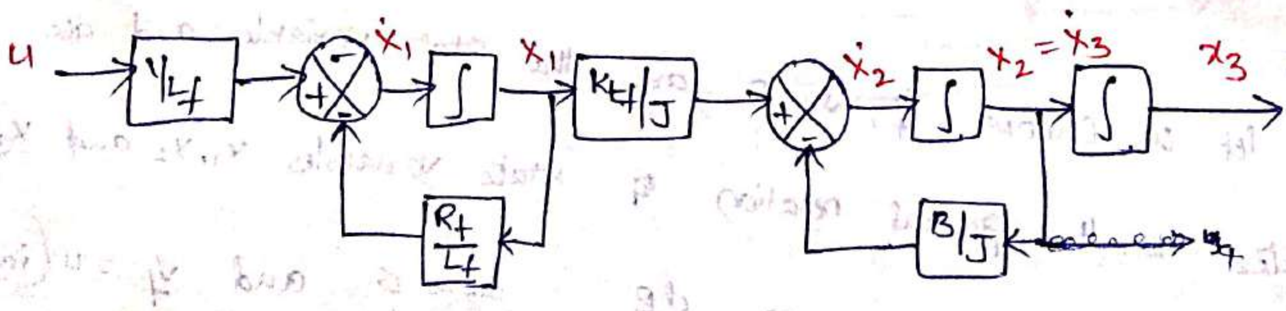
$$\frac{dx_3}{dt} = \frac{d\theta}{dt}$$

$$\Rightarrow \dot{x}_3 = x_2. \quad — ⑦$$

Using the ⑤, ⑥ and ⑦ equations, the state Model is written as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_f}{L_f} & 0 & 0 \\ \frac{K_{tf}}{J} & -\frac{B}{J} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_f} \\ 0 \\ 0 \end{bmatrix} u.$$

$$\dot{X} = A \quad X + B \quad U.$$



* Transfer function of State Model :-

The State model Consists of State equation and output Equation.

State equation $\Rightarrow \dot{x} = Ax + BU$

$$\Rightarrow \dot{x}(t) = Ax(t) + BU(t). \quad \text{—— (1)}$$

Output equation $\Rightarrow y(t) = Cx(t) + DU(t).$ —— (2)

On taking the laplace Transform for equation (1)

$$s\,X(s) = A\,X(s) + B\,U(s).$$

$$\Rightarrow s\,X(s) - A\,X(s) = B\,U(s).$$

$$\Rightarrow X(s)\,[sI - A] = B\,U(s). \quad \text{—— (A)}$$

from (2) $\Rightarrow Y(s) = C\,X(s) + D\,U(s).$ —— (3)

Now, Substitute $X(s)$ from (A) in (3)

$$\Rightarrow Y(s) = C\,[sI-A]^{-1} B\,U(s) + D\,U(s).$$

$$= U(s)\left[C\,[sI-A]^{-1}B + D\right]$$

$$\therefore \text{The } \boxed{T.F = \frac{Y(s)}{U(s)} = C\,[sI-A]^{-1}B + D.}$$

* State Space Representation Using phase Variables :-

The State Model using phase Variables can be easily determined if the System model is already known in the differential Equation (or) transfer function form. There are 3 methods of modelling a System using phase variables and they are explained in the following Cases.

Method-1

Consider the following n^{th} order differential equation relating the output to the input of a System.

$$\overset{n}{\dot{y}} + a_1 \overset{(n-1)}{\dot{y}} + \dots\dots a_{n-2}\ddot{y} + a_{n-1}\dot{y} + a_n y = bu.$$

By choosing the output y and their derivatives as State Variables, we get,

$$x_1 = y$$
$$x_2 = \dot{y}$$
$$x_3 = \ddot{y}$$
$$\vdots$$
$$x_n = \overset{(n-1)}{\dot{y}} \Rightarrow \dot{x}_n = \overset{n}{\dot{y}}$$

Now, on substituting the state variables in the differential equation,

$$\dot{x}_n + a_1 x_n + a_2 x_{n-1} + \dots + a_{n-2} x_3 + a_{n-1} x_2 + a_n x_1 = bu.$$

$$\therefore \dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \dots\dots - a_2 x_{n-1} - a_1 x_n + bu.$$

The State equations are

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = x_3$$
$$\vdots$$
$$\dot{x}_{n-1} = x_n$$
$$\dot{x}_n = -a_n x_1 - a_{n-1} x_2 - a_{n-2} x_3 - \dots\dots - a_2 x_{n-1} - a_1 x_n + bu.$$

On arranging the above equations in the Matrix form we gets

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots\dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots\dots & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & -a_{n-3} & \dots & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ +b \end{bmatrix} u$$

Here, the Matrix A (System matrix) has a very special form. It has all 1's in the upper off-diagonal, its last row is comprised of the negative of the coefficients of the original differential equation and all other elements are zero. This form of Matrix A is known as "BUCH FORM" (or) "COMPANION FORM."

we have assumed $y = x_1$

$$\therefore Y = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix}$$

Method-2

Consider the following nth order differential equation governing the output $y(t)$ to the input $u(t)$ of a system

$$\overset{n}{y} + a_1 \overset{(n-1)}{y} + \cdots + a_{n-1}\dot{y} + a_n y = b_0 \overset{m}{u} + b_1 \overset{(m-1)}{u} + \cdots b_{m-1}\dot{u} + b_m u$$

let $n = m = 3$.

$$\Rightarrow \dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3 y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3 u$$

Now, taking the laplace transform we get,

$$\Rightarrow s^3 Y(s) + a_1 s^2 Y(s) + a_2 s Y(s) + a_3 Y(s) = b_0 s^3 U(s) + b_1 s^2 U(s) + b_2 s U(s) + b_3 U(s)$$

$$\Rightarrow Y(s)\left(s^3 + a_1 s^2 + a_2 s + a_3\right) = U(s)\left(b_0 s^3 + b_1 s^2 + b_2 s + b_3\right)$$

$$\text{Now, } \frac{Y(s)}{U(s)} = \frac{b_0 s^3 + b_1 s^2 + b_2 s + b_3}{s^3 + a_1 s^2 + a_2 s + a_3}$$

$$= \frac{\cancel{s^3}\left[b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}\right]}{\cancel{s^3}\left(1 + \frac{a_1}{s} + \frac{a_2}{s^2} + \frac{a_3}{s^3}\right)} = \frac{b_0 + \frac{b_1}{s} + \frac{b_2}{s^2} + \frac{b_3}{s^3}}{1 - \left(-\frac{a_1}{s} - \frac{a_2}{s^2} - \frac{a_3}{s^3}\right)}$$

The transfer function of a system with four forward paths and three feedback loops is given by,

$$\boxed{T(s) = \frac{P_1 + P_2 + P_3 + P_4}{1 - (P_{11} + P_{12} + P_{13})}}$$

which is similar to the Mason's gain formula.

Here, P_1, P_2, P_3, P_4 are the forward paths and is given by,

$$P_1 = b_0, \quad P_2 = b_1/s, \quad P_3 = b_2/s^2 + , \quad P_4 = b_3/s^3.$$

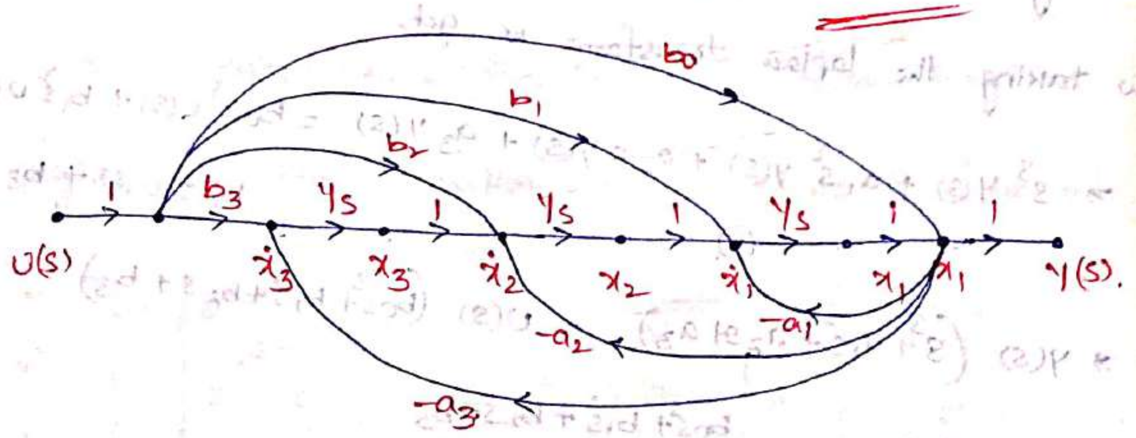
and P_{11}, P_{12}, P_{13} denotes the touching loops each other is given by,

$$P_{11} = \frac{-a_1}{s}, \quad P_{12} = \frac{-a_2}{s^2}, \quad P_{13} = \frac{-a_3}{s^3}.$$

Hence for the transfer function above, a signal flow graph can be constructed as shown below.

Now, let assign state variables at the output of each integrator in the in the signal flow graph. Hence at the input of each integrator, the first derivative of the state variable will be available. The state equations are formed by summing all the incoming signals to the nodes.,

(Direct Decomposition Method)



At node $\dot{x}_1$ $\Rightarrow$ $\dot{x}_1 = -a_1(x_1 + b_0 u) + x_2 + b_1 u$

$$= -a_1 x_1 + x_2 + (b_1 - a_1 b_0)u \quad \text{——— (1)}$$

At node $\dot{x}_2$

$$\Rightarrow \dot{x}_2 = -a_2(x_1+b_0u) + x_3 + b_2u$$

$$\Rightarrow \dot{x}_2 = -a_2x_1 + x_3 + (b_2 - a_2b_0)u \quad (2)$$

At node $\dot{x}_3$

$$\Rightarrow \dot{x}_3 = -a_3(x_1+b_0u) + b_3u$$

$$\therefore \dot{x}_3 = -a_3x_1 + (b_3 - a_3b_0)u. \quad (3)$$

and the output equation is given by the sum of the incoming signals to output node.

$$\therefore y = x_1 + b_0u. \quad (4)$$

$\therefore$ The state model can be Represented as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -a_1 & 1 & 0 \\ -a_2 & 0 & 1 \\ -a_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} b_1 - a_1b_0 \\ b_2 - a_2b_0 \\ b_3 - a_3b_0 \end{bmatrix} u$$

and
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + b_0u.$$

Method-3.

Consider the differential equation governing the output $y(t)$ to the input $u(t)$ of a system.

$$\dddot{y} + a_1\ddot{y} + a_2\dot{y} + a_3y = b_0\dddot{u} + b_1\ddot{u} + b_2\dot{u} + b_3u$$

Taking the laplace transform for above equation

$$Y(s)s^3 + a_1s^2Y(s) + a_2sY(s) + a_3Y(s) = [b_0s^3 + b_1s^2 + b_2s + b_3]U(s).$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{b_0s^3 + b_1s^2 + b_2s + b_3}{s^3 + a_1s^2 + a_2s + a_3}$$

let
$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{X_1(s)} \times \frac{X_1(s)}{U(s)}.$$

where, $\frac{X_1(S)}{U(S)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$ and, $\frac{Y(S)}{X_1(S)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$.

Now, $\frac{X_1(S)}{U(S)} = \frac{1}{s^3 + a_1 s^2 + a_2 s + a_3}$

$$X_1(S)\left[s^3 + a_1 s^2 + a_2 s + a_3\right] = U(S).$$

$$s^3 X_1(S) + a_1 s^2 X_1(S) + a_2 s\, X(S) + a_3 X(S) = U(S).$$

on taking inverse laplace transform,

$$\dddot{x}_1 + a_1 \ddot{x}_1 + a_2 \dot{x}_1 + a_3 x_1 = u \quad \text{(A)}$$

Let the state variables be, x_1, x_2, x_3.

where, $x_2 = \dot{x}_1$, $x_3 = \ddot{x}_1 = \dot{x}_2$, $\dot{x}_3 = \dddot{x}_1$

on substituting state variables in equation (A)

$$\text{(A)} \Rightarrow \dot{x}_3 + a_1 x_3 + a_2 x_2 + a_3 x_1 = u$$

$$\Rightarrow \dot{x}_3 = -a_3 x_1 + (-a_2 x_2) - a_1 x_3 + u. \quad \text{(C)}$$

∴ The state equations are :

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= -(a_3 x_1 + a_2 x_2 + a_1 x_3) + u \end{aligned}\right\} \quad \text{(B)}$$

Now $\frac{Y(S)}{X_1(S)} = b_0 s^3 + b_1 s^2 + b_2 s + b_3$

$$\Rightarrow Y(S) = b_0 s^3 X_1(S) + b_1 s^2 X_1(S) + b_2 s X_1(S) + b_3 X_1(S).$$

on taking inverse laplace transform,

$$\Rightarrow y = b_0 \dddot{x}_1 + b_1 \ddot{x}_1 + b_2 \dot{x}_1 + b_3 x_1$$

$$= b_0 \dot{x}_3 + b_1 x_3 + b_2 x_2 + b_3 x_1$$

Now substitute eq (C) in above equation we get,

$$y = b_0(-a_3x_1 - a_2x_2 - a_1x_3 + u) + b_1x_3 + b_2x_2 + b_3x_1$$

$$= (b_3 - a_3b_0)x_1 + (b_2 - a_2b_0)x_2 + (b_1 - a_1b_0)x_3 + b_0u. \quad \text{(D)}$$

Now, the state model from the equations (B) and (D) is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} (b_3 - a_3b_0) & (b_2 - a_2b_0) & (b_1 - a_1b_0) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + [b_0]u.$$

* **Problems:**

1. Construct the state model for a system characterized by a differential equation. $\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$

Sol: given DE $= \frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y + u = 0.$

let us choose 'y' and their derivatives as state variables.

let $y = x_1.$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \ddot{x}_1 = \dot{x}_2 = x_3$

$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3.$

Now substitute the above variables in D.E

$\Rightarrow \dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + u = 0$

$\Rightarrow \dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - u$

Therefore, the state equations are,

$\dot{x}_1 = x_2$

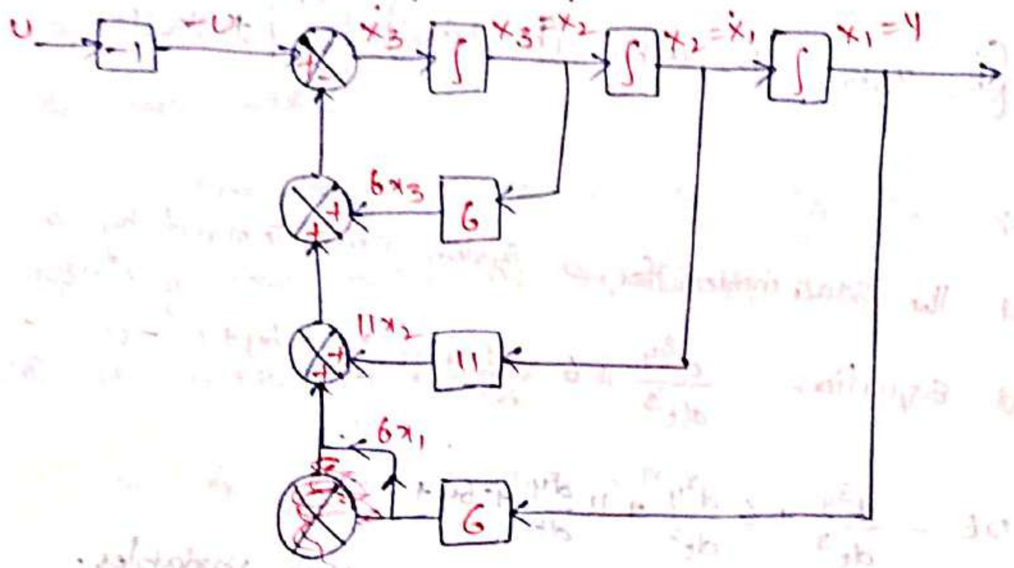
$\dot{x}_2 = x_3$

$\dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - u$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} u.$$

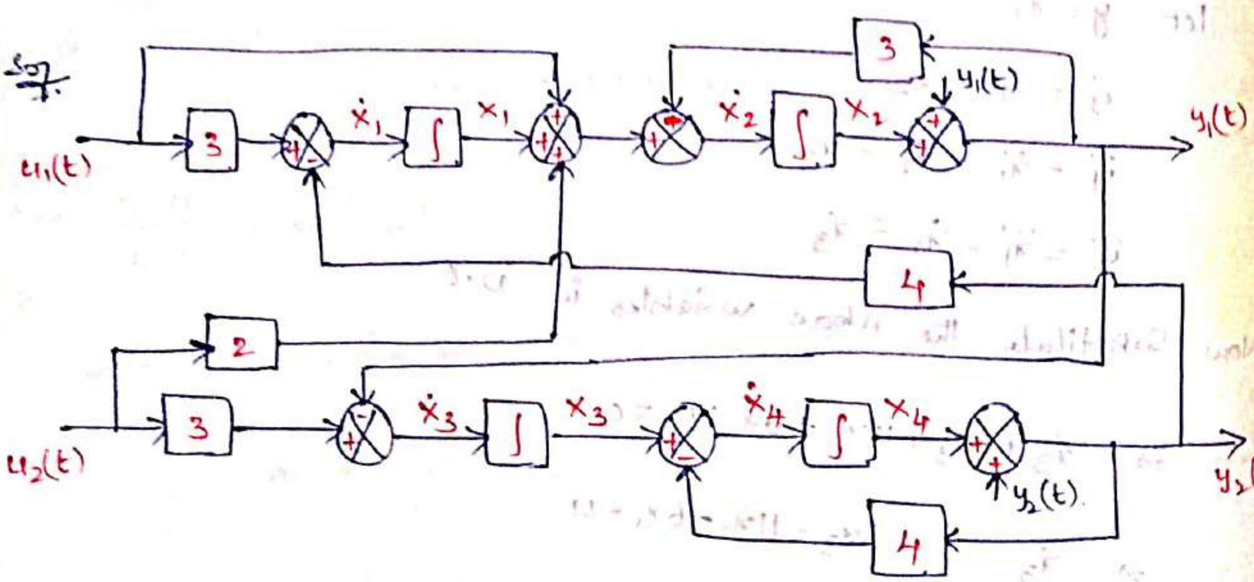
Here, output $y = x_1$

$$\therefore y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$



② The state Diagram of a System is shown below. Assign state variables and obtain the state Model of the system.

Sol.



Sol. Since there are 4-integrators in the state diagram, we can assign 4 state variables. The state variables can be assigned to the output of the integrators. Hence, at the input of integrator, the first derivative of the state variable will be available. The state equations are formed by summing all the incoming signals to the input of the integrator and equating to the corresponding first derivative of the system

$$\dot{x}_1 = -4x_4 + 3u_1$$

$$\dot{x}_2 = x_1 - 3x_2 + u_1 + 2u_2$$

$$\dot{x}_3 = -x_2 + 3u_2$$

$$\dot{x}_4 = x_3 - 4x_4.$$

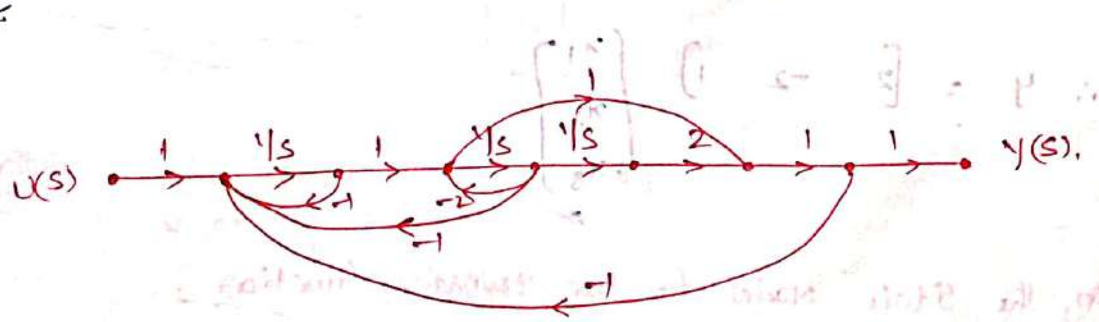
The output equations are $y_1 = x_2$, $y_2 = x_4$.

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix} = \begin{bmatrix}0 & 0 & 0 & -4\\ 1 & -3 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}3 & 0\\ 1 & 2\\ 0 & 3\\ 0 & 0\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

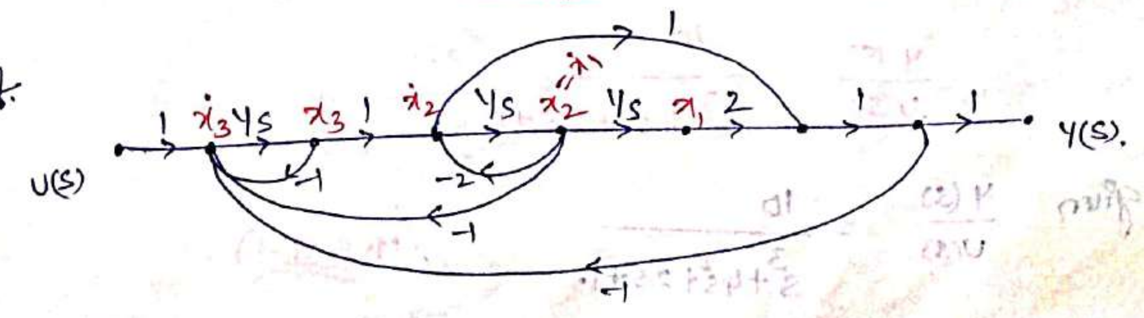
$$\begin{bmatrix}y_1\\ y_2\end{bmatrix} = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}$$

3. Obtain the state Model for the signal flow graph below. ?

Sol.



Sol.



On adding the signals coming to node-5.

$$\dot{x}_1 = x_2 \quad \text{---} (1)$$

Similarly,

$$\dot{x}_2 = -2x_2 + x_3 \quad \text{---} (2)$$

$$\dot{x}_3 = -(\dot{x}_2 + 2x_1) - x_2 - x_3 + 4$$

$$= -\dot{x}_2 - 2x_1 - x_2 - x_3 + 4$$

Now, Substitute $\dot{x}_2 = -2x_2 + x_3$ in above eqn

$$\Rightarrow \dot{x}_3 = -(-2x_2 + x_3) - 2x_1 - x_2 - x_3 + 4$$

$$= 2x_2 - x_3 - 2x_1 - x_2 - x_3 + 4$$

$$\therefore \dot{x}_3 = -2x_1 + x_2 - 2x_3 + 4 \quad \text{---} (3)$$

from the (1), (2), (3) equations, the state model is obtained as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} 4$$

The output Equation is, $y = 2x_1 + \dot{x}_2$

Substitute $\dot{x}_2 = -2x_2 + x_3$

$$\Rightarrow y = 2x_1 - 2x_2 + x_3.$$

$$\therefore y = \begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

4. Obtain the State Model for the Transfer function

$$\frac{Y(S)}{U(S)} = \frac{10}{S^3 + 4S^2 + 2S + 1}$$

Sol. given $\dfrac{Y(S)}{U(S)} = \dfrac{10}{S^3 + 4S^2 + 2S + 1}$. (Method-1)

$\Rightarrow Y(s)\left[s^3+4s^2+2s+1\right] = 10\,U(s).$

$\Rightarrow s^3 Y(s) + 4s^2 Y(s) + 2s\,Y(s) + Y(s) = 10\,U(s).$

$\rightarrow$ on taking inverse laplace transform we get,

$\Rightarrow \dddot{y} + 4\ddot{y} + 2\dot{y} + y = 10u$ — ①

Now, $y = x_1$

$\dot{y} = \dot{x}_1 = x_2$

$\ddot{y} = \ddot{x}_1 = \dot{x}_2 = x_3$

$\dddot{y} = \dddot{x}_1 = \ddot{x}_2 = \dot{x}_3$

① $\Rightarrow \dot{x}_3 + 4x_3 + 2x_2 + x_1 = 10u$

$\Rightarrow \dot{x}_3 = 10u - 4x_3 - 2x_2 - x_1$

$\therefore$ Now, the state equations are,

$\dot{x}_1 = x_2,\quad \dot{x}_2 = x_3,\quad \dot{x}_3 = 10u - 4x_3 - 2x_2 - x_1.$

output Equation $y = x_1.$

$$\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -2 & -4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 10\end{bmatrix} u$$

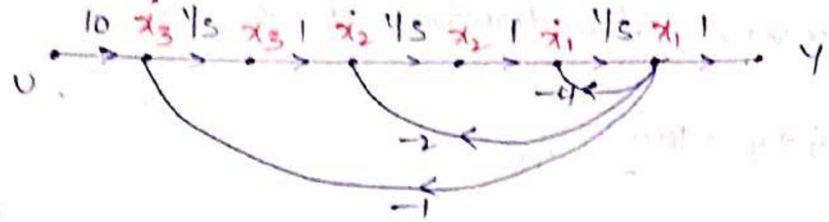
$$y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}.$$

Method-2

$$\frac{Y(s)}{U(s)} = \frac{10}{s^3+4s^2+2s+1}$$

$$= \frac{10/s^3}{1-\left(\frac{-4}{s}-\frac{2}{s^2}-\frac{1}{s^3}\right)}$$

The signal flow graph from a transfer function Consists of one forward path and three individual loops gains $-4/s$, $-2/s^2$, $-1/s^3$.



The state equations from the above signal flow graph.

$$\dot{x}_1 = -4x_1 + x_2$$

$$\dot{x}_2 = -2x_1 + x_3$$

$$\dot{x}_3 = -x_1 + 10u.$$

output equation $y = x_1$.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -2 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 10 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Method-3:

given $\dfrac{Y(s)}{U(s)} = \dfrac{10}{s^3+4s^2+2s+1}$

$\Rightarrow \dfrac{Y(s)}{X_1(s)} \times \dfrac{X_1(s)}{U(s)} = \dfrac{10}{s^3+4s^2+2s+1}.$

$\Rightarrow \dfrac{Y(s)}{X_1(s)} = 10, \quad \dfrac{X_1(s)}{U(s)} = \dfrac{1}{s^3+4s^2+2s+1}$

$\Rightarrow Y(s) = 10\, X_1(s), \quad X_1(s) \left[s^3+4s^2+2s+1\right] = U(s)$

$\Rightarrow$ on taking inverse laplace transform we get,

$\Rightarrow y = 10x_1. \quad , \quad \dddot{x}_1 + 4\ddot{x}_1 + 2\dot{x}_1 + x_1 = u$

$$\dot{x}_3 + 4x_3 + 2x_2 + x_1 = u$$

Now, the state equations are,

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3.$

$\dot{x}_3 = -4x_3 - 2x_2 - x_1 + u$

and output equation $y = 10x_1.$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

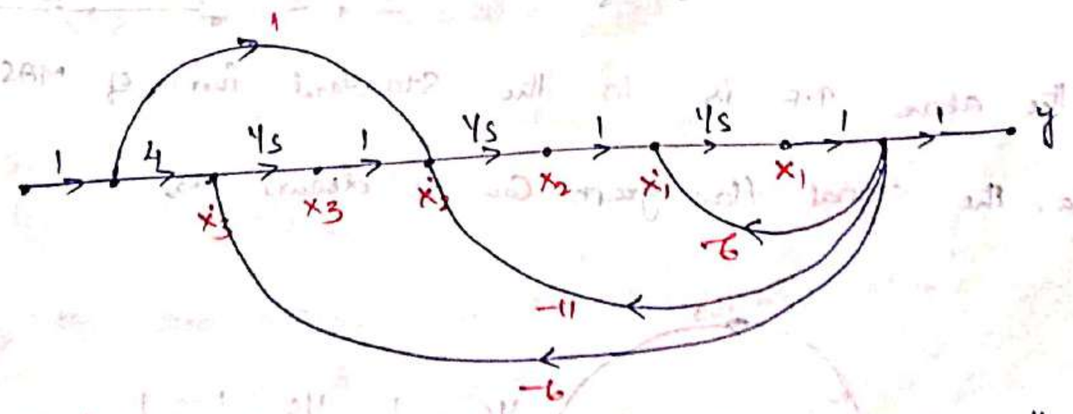
5. Find the State Model for the System with T.F $= \dfrac{s+4}{s^3+6s^2+11s+6}$

Sol. given T.F $= \dfrac{Y(s)}{U(s)} = \dfrac{s+4}{s^3+6s^2+11s+6}.$

Now, Divide the denominator and Numerator with s^3

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{1}{s^2} + \frac{4}{s^3}}{1 + \frac{6}{s} + \frac{11}{s^2} + \frac{6}{s^3}} = \frac{\frac{1}{s^2} + \frac{4}{s^3}}{1 - \left[-\frac{6}{s} - \frac{11}{s^2} - \frac{6}{s^3}\right]}.$$

∴ From the above equation we have two forward paths and the terms in denominator represents the three loops as shown in the figure



Now, the differential equations should be attained from the above signal flow graph.

The state equations are Represented from the signal flow graph, i.e.,

$$\dot{x}_1 = -6x_1 + x_2$$

$$\dot{x}_2 = -11x_1 + x_3 + u$$

$$\dot{x}_3 = -6x_1 + 4u.$$

From the above state equations the model will be written as,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}-6 & 1 & 0\\ -11 & 0 & 1\\ -6 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 1\\ 4\end{bmatrix}u.$$

and $y = x_1 \Rightarrow y = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$.

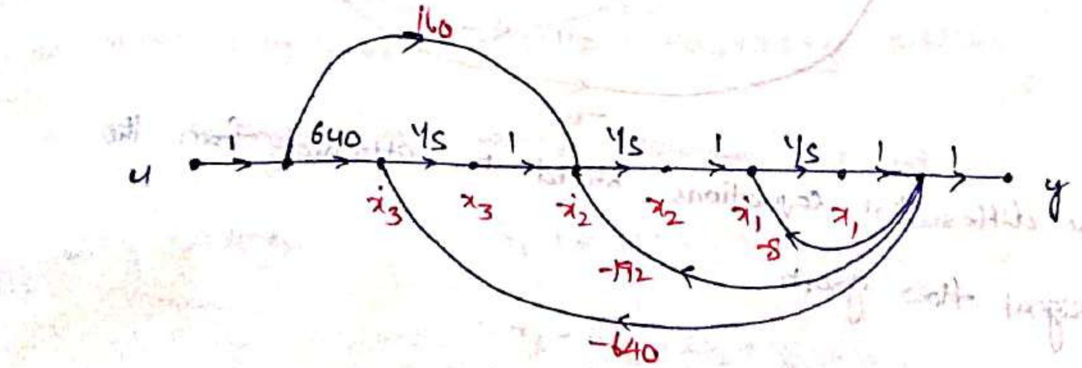
6. The closed loop T.F is given by, $\dfrac{Y(S)}{U(S)} = \dfrac{160(S+4)}{S^3+8S^2+192S+640}$. Obtain the state model using signal flow graph.

Sol. given, $T.F = \dfrac{Y(S)}{U(S)} = \dfrac{160(S+4)}{S^3+8S^2+192S+640}$.

Now, divide the numerator and denominator with s^3.

$$\Rightarrow \frac{Y(S)}{U(S)} = \frac{\frac{160}{s^2}+\frac{4\times160}{s^3}}{1+\frac{8}{s}+\frac{192}{s^2}+\frac{640}{s^3}} = \frac{\frac{160}{s^2}+\frac{640}{s^3}}{1-\left(-\frac{8}{s}-\frac{192}{s^2}-\frac{640}{s^3}\right)}.$$

Now, the above T.F is in the standard form of MASON's gain formula, the signal flow graph can be drawn as,



let x_1, x_2, x_3 be the state variables.

The state equation and the output equations are written from the signal flow graph is given by,

$$\dot{x}_1 = -8x_1 + x_2$$

$$\dot{x}_2 = -192x_1 + x_3 + 160u$$

$$\dot{x}_3 = -640x_1 + 640u.$$

and $y = x_1$ ~~+160u~~

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -8 & 1 & 0 \\ -192 & 0 & 1 \\ -640 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 160 \\ 640 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0, & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

7. $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\ddot{u} + 6\dot{u} + 5u$

Sol: given that $\dddot{y} + 4\ddot{y} + 5\dot{y} + 2y = 2\ddot{u} + 6\dot{u} + 5u.$

$$\Rightarrow \frac{d^3y}{dt^3} + 4\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 2y = 2\frac{d^2u}{dt^2} + 6\frac{du}{dt} + 5u.$$

Now, apply inverse laplace transform,

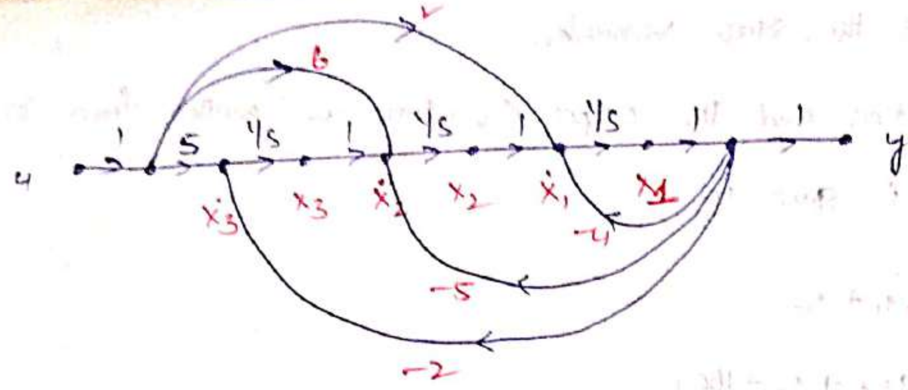
$$\Rightarrow s^3Y(s) + 4s^2Y(s) + 5sY(s) + 2Y(s) = 2s^2U(s) + 6sU(s) + 5U(s)$$

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{2s^2 + 6s + 5}{s^3 + 4s^2 + 5s + 2}$$

Dividing the denominator and Numerator with s^3.

$$\Rightarrow \frac{Y(s)}{U(s)} = \frac{\frac{2}{s} + \frac{6}{s^2} + \frac{5}{s^3}}{1 - \left(\frac{-4}{s} - \frac{5}{s^2} - \frac{2}{s^3}\right)}.$$

From the above transfer function, obtain the signal flow graph.



From the above flow graph.

The state equations are $\dot{x}_1 = -4x_1 + x_2 + 2u$

$$\dot{x}_2 = -5x_1 + x_3 + 6u$$

$$\dot{x}_3 = -2x_1 + 5u$$

and $y = x_1$

$$\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -4 & 1 & 0 \\ -5 & 0 & 1 \\ -2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} u.$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

8. A State Model is characterised by.

$$\dot{x} = \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

find. T.F

Sol From the above State Model

$$A = \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Now, the transfer function $= C\,[sI - A]^{-1} B.$

$$\left|[sI-A]\right| = s \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & -4 & -1 \\ -1 & -6 & -2 \\ -1 & -2 & -3 \end{bmatrix} = \begin{bmatrix} s+1 & 4 & 1 \\ 1 & s+6 & 2 \\ 1 & 2 & s+3 \end{bmatrix}$$

$$|sI-A| = s^3+10s^2+18s+6.$$

$$[sI-A]^{-1} = \frac{Adj\,[sI-A]}{|sI-A|}.$$

Co-factors, $(sI-A) = \begin{bmatrix} s+1 & 4 & 1 \\ 1 & s+6 & 2 \\ 1 & 2 & s+3 \end{bmatrix}$

$$C_{11} = s+6\,(s+3) - 4 \qquad , \qquad C_{12} = 1\,(s+3) - 2\times 1$$
$$= s^2+9s+14 \qquad\qquad = s+1$$

$$C_{13} = 2-(s+6)$$
$$= -s-4$$

Similarly, $C_{21} = 4s+10$ and $C_{31} = -s+2$

$$C_{22} = s^2+4s+2 \qquad C_{32} = 2s+1$$

$$C_{23} = 2s-2 \qquad C_{33} = s^2+7s+2$$

~~Now~~, ~~Co-factor~~ Matrix =

$$\text{Now, } Adj\,[sI-A] = \begin{bmatrix} +(s^2+9s+14) & -(4s+10) & +(-s+2) \\ -(s+1) & +(s^2+4s+2) & -(2s+1) \\ +(-s-4) & -(2s-2) & +(s^2+7s+2) \end{bmatrix}$$

$$\text{Now, } T.F = C \cdot [sI-A]^{-1} B$$

$$= \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \frac{1}{s^3+10s^2+18s+6} \begin{bmatrix} s^2+9s+14 & -(4s+10) & (-s+2) \\ -(s+1) & (s^2+4s+2) & -(2s+1) \\ (-s-4) & -(2s-2) & s^2+7s+2 \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^3+10s^2+18s+6} \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} -5s-8 \\ s^2+2s+1 \\ s^2+5s+4 \end{bmatrix}_{3\times 1} \rightarrow (1\times 1)$$

$$\therefore T.F = \frac{2s^2+2s+(-3)}{s^3+10s^2+18s+6}$$

9. given, $A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Find the characteristic equation of the system and then find the roots.

Sol. As we know that the characteristic equation will be taken or considered as the Denominator term of the obtained transfer function. From the T.F of the state Model

$$\Rightarrow T.F = C.[SI-A]^{-1}B + D \quad (\text{or}) \quad C[SI-A]^{-1}B$$

$$= C \frac{Adj(SI-A)}{|SI-A|} B.$$

Now from the above Transfer function the characteristic equation is

$$|SI-A| = \left| S\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \right|$$

$$= \begin{vmatrix} S-1 & -2 & -1 \\ 0 & S-1 & -3 \\ -1 & -1 & S-1 \end{vmatrix}$$

$$= S-1\left[(S-1)^2-3\right] + 2(-3) - 1(S-1)$$

$$= S-1\,(S^2-2S+1-3) - 6 - S+1$$

$$= S^3-2S^2-2S-S^2+2S+2-S-5$$

$$\therefore |SI-A| = S^3-3S^2-S+(-3).$$

So, the characteristic equation is, $S^3-3S^2-S-3=0$.

From the above equation the roots of 'S' are

$3.525, \; -0.2625 \pm j\,0.884$

10. $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u, \quad y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

find the T.F

Sol: given $A = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$

$\therefore$ T.F $= C\,[sI-A]^{-1} B$

$$|(sI-A)| = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 1 \\ -3 & -4 & -5 \end{bmatrix}$$

$$= \left| \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix} \right|$$

$$= s^3 + 10s^2 + 35s + 41$$

$$[sI-A]^{-1} = \frac{\operatorname{adj}(sI-A)}{|sI-A|}$$

$$\frac{\operatorname{adj}(sI-A)}{|sI-A|} = \operatorname{adj} \begin{bmatrix} s+2 & -1 & 0 \\ 0 & s+3 & -1 \\ 3 & 4 & s+5 \end{bmatrix}$$

$$= \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -(3s+9) & -(4s+11) & s^2+5s+6 \end{bmatrix} \Big/ s^3+10s^2+35s+41$$

$\therefore$ The T.F $= C\,[sI-A]^{-1} B$

$$= \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \frac{1}{s^3+10s^2+35s+41} \begin{bmatrix} s^2+8s+19 & s+5 & 1 \\ -3 & s^2+7s+10 & s+2 \\ -(3s+9) & -(4s+11) & s^2+5s+6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore \text{The T.F} = \boxed{\frac{s+2}{s^3+10s^2+35s+41}}$$

11. Diagnolize the matrix $A = \begin{bmatrix} 8 & -6 & +2 \\ -6 & +7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

Sol. given, $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Now, $|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \right| = 0$

$$= \begin{vmatrix} \lambda-8 & 6 & -2 \\ 6 & \lambda-7 & 4 \\ -2 & 4 & \lambda-3 \end{vmatrix} = 0$$

$\Rightarrow \lambda-8\,[(\lambda-7)(\lambda-3)-16] - 6\,(6\lambda-18+8) - 2\,[24+2(\lambda-7)] = 0$

$\Rightarrow (\lambda-8)(\lambda^2-10\lambda+5) - 6(6\lambda-10) - 2\,(2\lambda+10) = 0$

$\Rightarrow \lambda^3 - 18\lambda^2 + 45\lambda + 0 = 0$

$\lambda = 0, 3, 15$

$[\lambda_1 I - A] = \begin{bmatrix} -8 & 6 & -2 \\ 6 & -7 & 4 \\ -2 & 4 & -3 \end{bmatrix}$ $(\lambda_1 = 0)$

$c_{11} = 21-16 = 5$

$c_{12} = -(-18+8) = 10$ $\therefore X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$c_{13} = 24-14 = 10.$

$[\lambda_2 I - A] = \begin{bmatrix} -5 & 6 & -2 \\ 6 & -4 & 4 \\ -2 & 4 & 0 \end{bmatrix}$ $(\lambda_2 = 3)$

$C_{11} = -16.$

$C_{12} = -(0+8)$
$= -8$

$$\therefore X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$$

$C_{13} = 24-8$
$= 16$

Similarly, $[\lambda_3 I - A] = \begin{bmatrix} 7 & 6 & -2 \\ 6 & 8 & +4 \\ -2 & 4 & 12 \end{bmatrix}$ $(\lambda_3 = 15)$

$C_{11} = 96-16$
$= 80$

$C_{12} = -(72+8)$
$= -80$

$$\therefore X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$$

$C_{13} = 24+16$
$= 40$

$\boxed{P = \text{Modal Matrix}}$

$$\therefore P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\text{Now, } P^{-1} = \frac{adj(P)}{|P|} = \begin{bmatrix} 0.111 & 0.22 & 0.22 \\ -0.22 & -0.11 & 0.22 \\ 0.22 & -0.22 & 0.11 \end{bmatrix}$$

Now, Diagonalization of a Matrix A is given by,

$$D = P^{-1}AP = \begin{bmatrix} 0.11 & 0.22 & 0.22 \\ -0.22 & -0.11 & 0.22 \\ 0.22 & -0.22 & 0.11 \end{bmatrix} \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2.97 & 0 \\ 0 & 0 & 14.85 \end{bmatrix} \cong \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

12. Diagonalize the matrix $A = \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix}$

Sol. Now, C.E $\Rightarrow |\lambda I - A| = 0$

$$\Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda-8 & 8 & 2 \\ -4 & \lambda+3 & 2 \\ -3 & +4 & \lambda-1 \end{vmatrix} = 0$$

$$\Rightarrow \lambda-8 \left[((\lambda+3)(\lambda-1)) - 8 \right] - 8(-4\lambda+4+6) + 2 \left[-16 + 3(\lambda+3) \right] = 0$$

$$\Rightarrow \lambda-8 \; (\lambda^2+2\lambda-3-8) - 8 \; (-4\lambda+10) + 2 \; (3\lambda-7) = 0$$

$$\Rightarrow \lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$\lambda = 3, 1, 2 \Rightarrow \lambda_1 = 1, \; \lambda_2 = 2, \; \lambda_3 = 3.$$

Now,

$$|\lambda_1 I - A| = \begin{bmatrix} -7 & 8 & 2 \\ -4 & 4 & 2 \\ -3 & 4 & 0 \end{bmatrix}$$

$c_{11} = -8, \quad c_{12} = -(0+6) = -6, \quad c_{13} = -16+12 = -4.$

$$\Rightarrow M_1 = \begin{pmatrix} -8 \\ -6 \\ -4 \end{pmatrix} = \begin{pmatrix} +4 \\ +3 \\ 2 \end{pmatrix}.$$

Similarly, $|\lambda_2 I - A| = \begin{pmatrix} -6 & 8 & 2 \\ -4 & 5 & 2 \\ -3 & 4 & 1 \end{pmatrix}$

$c_{11} = -3, \quad c_{12} = -(-4+6) = -2, \quad c_{13} = -16+15 = -1.$

$$\therefore M_2 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

$|\lambda_3 I - A| = 0$

$$= \begin{bmatrix} -5 & 8 & 2 \\ -4 & 6 & 2 \\ -3 & 4 & 2 \end{bmatrix}$$

$c_{11} = 12-8 = 4$

$c_{12} = -(-8+6) = 2$

$c_{13} = -16+18 = 2$

$$\therefore M_3 = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$\therefore P = [M_1 \; M_2 \; M_3] = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$\therefore \text{Now, } P^{-1} = \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$\therefore$ Now, Diagnolization of Matrix A

$$= P^{-1} A P$$

$$= \begin{pmatrix} -1 & 1 & +1 \\ 1 & 0 & -2 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 1 & 1 \\ 2 & 0 & -4 \\ 3 & -6 & 3 \end{pmatrix} \begin{pmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Diagnolized Matrix should contain their Eigen values in as diagonal elements and Remaining elements will be zero.

13. Diagonalize the Matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$.

Sol. given, $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$|\lambda I - A| = 0 \Rightarrow \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

$\Rightarrow \begin{vmatrix} \lambda-1 & -1 & -3 \\ -1 & \lambda-5 & -1 \\ -3 & -1 & \lambda-1 \end{vmatrix} = 0$

$\Rightarrow \lambda-1 \left[(\lambda-5)(\lambda-1) - 1\right] + 1 \left(-1(\lambda-1) - 3\right) - (+3) \left[1 + 3(\lambda-5)\right] = 0$

$\Rightarrow (\lambda-1)(\lambda^2 - 5\lambda - \lambda + 5 - 1) + (-\lambda + 1 - 3) + 3(3\lambda - 14) = 0$

$\Rightarrow (\lambda-1)(\lambda^2 - 6\lambda + 4) + (-\lambda - 2) - 9\lambda + 42 = 0$

$\Rightarrow \lambda^3 - 6\lambda^2 + 4\lambda - \lambda^2 + 6\lambda - 4 - \lambda - 2 - 9\lambda + 42 = 0$

$\Rightarrow \lambda^3 - 7\lambda^2 + 0\lambda + 36 = 0$

$\therefore \lambda = -2, 3, 6.$

$[\lambda_1 I - A] \Rightarrow \begin{bmatrix} -3 & -1 & -3 \\ -1 & -7 & -1 \\ -3 & -1 & -3 \end{bmatrix}$

$C_{11} = 21 - 1 = 20$

$C_{12} = -[3-3] = 0$ $\qquad \therefore X_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

$C_{13} = 1 - 21 = -20$

$(\lambda_2 I - A) \Rightarrow \begin{bmatrix} 2 & -1 & -3 \\ -1 & -2 & -1 \\ -3 & -1 & 2 \end{bmatrix}$

$c_{11} = -4-1 = -5$

$c_{12} = -[2-3] = 5$

$c_{13} = 1-6 = -5$

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Similarly, $[\lambda_3 I - A] \Rightarrow \begin{bmatrix} 5 & -1 & -3 \\ -1 & 1 & -1 \\ -3 & -1 & 5 \end{bmatrix}$

$c_{11} = 5-1 = 4$

$c_{12} = -(-5-3) = +8$

$c_{13} = 1+3 = 4$

$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

Now, $P = [X_1 \quad X_2 \quad X_3]$

$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}.$

$P^{-1} = \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix}$

Now, Diagonalization $D = P^{-1}AP$

$\Rightarrow D = \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2.97 & 0 \\ 0 & -0.006 & 5.95 \end{bmatrix} \cong \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

Powers of a Matrix.

As we know that $D = P^{-1}AP$.

Now, squaring on both sides we get,

$$D^2 = (P^{-1}AP)^2$$
$$= (P^{-1}AP)(P^{-1}AP)$$
$$= (P^{-1}A)(PP^{-1})(AP)$$
$$= P^{-1}A(I)(AP)$$
$$= (P^{-1}A)(AP)$$

$\therefore \boxed{D^2 = P^{-1}A^2P}$

Similarly, $D^3 = P^{-1}A^3P$.

in general $\Rightarrow \boxed{D^n = P^{-1}A^nP}$

$\Rightarrow D^n = P^{-1}A^nP$

$\Rightarrow \boxed{A^n = PD^nP^{-1}}$

4. Now, find A^4 for above problem for diagonalized Matrix A

Sol. $D^4 = P^{-1}A^4P$.

$$\Rightarrow A^4 = P^{-1}D^4P$$

$$= \begin{bmatrix} 0.5 & 0 & -0.5 \\ -0.33 & 0.33 & -0.33 \\ 0.166 & 0.33 & 0.166 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 1296 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$\Rightarrow A^4 = \begin{bmatrix} 251 & 405 & 235 \\ 405 & 891 & 405 \\ 235 & 405 & 251 \end{bmatrix}$$

15. $A = \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Show that

$P^{-1}AP$ is a diagonal matrix.

Sol: given, $A = \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Now, $P^{-1} = \dfrac{\text{adj } P}{|P|}$

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now, $D = P^{-1} A P$.

$$= \begin{bmatrix} 0.5 & -0.866 & 0 \\ 0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.5 & -0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & 0.866 & 0 \\ -0.866 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3.8\times10^{-5} & 0 \\ -3.85\times10^{-5} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\simeq \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

* Solution of State Equations without input or Excitation :-

Consider a first order differential equation, with initial Condition $x(0) = x_0$.

$$\frac{dx}{dt} = ax \quad ; \quad x(0) = x_0.$$

on Re-arranging the above equation.

$$\frac{dx}{x} = a \cdot dt.$$

Now, on integrating the above equation,

neglect, $\int \frac{dx}{x} = \int a\,dt$

$\Rightarrow \log x = at + C$

$\Rightarrow x = e^{(at+C)}$

$\therefore x = e^{at} \cdot e^{C}$ —— ①

$\left(\because x(0) = e^{a \cdot 0} \cdot e^{C} \right.$
$\Rightarrow x_0 = 1 \cdot e^{C}$
$\left. \Rightarrow e^{C} = x_0 . \right)$

When, $t=0$ from the above equation, we get,

$x = x(0) = e^{C}$

given that, $x(0) = x_0$ $\therefore e^{C} = x_0$. —— ②

Now Substituting ② in ① we get,

$\boxed{x = e^{at} \cdot x_0 .}$ —— ③

we know that, $e^{x} = \left[1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n \right]$ —— ④

from eq. ③ and ④

$\Rightarrow X = e^{at}X_0 = \left[1 + at + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots + \frac{1}{n!}(at)^{n} + \cdots \right] X_0 .$ —— ⑤

Now, Consider the state equation with out input vector.

$\dot{x}(t) = Ax(t) \quad ; \quad x(0) = x_0$ —— Ⓐ

where, $x(0)$ is the initial condition vector;

Now, the solution of the eqn ⑤ can be written in the matrix equation as follows,

$x(t) = A_0 + A_1 t + A_2 t^2 + \cdots + A_i t^i + \cdots$ —— ⑥

where, A_0, A_1, A_2, A_3 are matrices.

Now, differentiate the equation ⑥ we get,

$\dot{x}(t) = A_1 + 2A_2 t + 3A_3 t^2 + \cdots + iA_i t^{(i-1)} \cdots$ —— ⑦

on Multiplying the above equation ⑥ with A on both sides we get,

$$\Rightarrow A\,x(t) = A\left[A_0 + A_1 t + A_2 t^2 + \cdots + iA_i t^{i-1} + \cdots\right] \quad (8)$$

from the equation (A)

$$\dot{x}(t) = A\,x(t).$$

∴ equate the equations (7) and (8) we get,

$$\boxed{A_1 = A A_0} \quad (9)$$

⇒ equating 't' coefficients

$$\Rightarrow 2A_2 = A A_1$$

$$\Rightarrow A_2 = \frac{1}{2} A A_1 \quad (10)$$

Put $A_1 = A A_0$ in (10)

$$\Rightarrow \boxed{A_2 = \frac{1}{2} A^2 A_0} \quad (11)$$

⇒ equating t^2 coefficients we gets

$$\Rightarrow 3A_3 = A A_2$$

$$A_3 = \frac{1}{3} A A_2 \quad (12)$$

Now, Substitute A_2 in (12) we gets

$$\Rightarrow A_3 = \frac{1}{3} A \frac{1}{2} A^2 A_0$$

$$= \frac{1}{6} A^3 A_0.$$

$$\therefore \boxed{A_3 = \frac{1}{3!} A^3 A_0}$$

Similarly, $A_4 = \frac{1}{4!} A^4 A_0$.

$$\therefore x(t) = \left[A_0 + A A_0 t + \frac{1}{2!} A^2 A_0 t^2 + \frac{1}{3!} t^3 A^3 A_0 + \cdots + \frac{1}{n!} t^n A^n A_0 + \cdots\right]$$

$$= \left[I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} A^3 t^3 + \cdots + \frac{1}{n!} A^n t^n + \cdots\right] A_0$$

where, $A_0 = X_0$. [∵ from eq. (6) when $t=0$]

$$\Rightarrow x(t) = \left[I + At + \frac{1}{2!} A^2 t^2 + \frac{1}{3!} (At)^3 + \cdots\right] X_0.$$

$$\therefore \boxed{X(t) = e^{At} X_0} \quad (13)$$

∴ The above equation is called Solution of State Equation.

and e^{At} is called state transition Matrix and is denoted by $\phi(t)$.

*** properties of State transition Matrix :-**

1. $\phi(0) = e^{A\times 0} = I$ (unit Matrix)

2. $\phi(t) = e^{At} = \left(e^{-At}\right)^{-1} = [\phi(-t)]^{-1}$

(or)

$\cancel{\phi^{-1}(t)} = \cancel{\phi(-t)}$. $\phi^{-1}(t) = \phi(-t)$.

3. $\phi(t_1+t_2) = e^{At_1}.e^{At_2} = \phi(t_1).\phi(t_2) = \phi(t_2).\phi(t_1)$.

*** Computation of State transition Matrix**

Method-1 : Computation of e^{At} using Matrix Exponential

Method - 2 : Computation of e^{At} using laplace transform

Method - 3 : " " " Canonical transformation.

Method - 4 : " " " Caley Hamilton Theorem.

*** Computation of e^{At} by Exponential Method**

As we know,

$$e^{\lambda t} = 1 + \lambda.t + \frac{1}{2!}\lambda^2 t^2 + \frac{1}{3!}(\lambda t)^3 + \frac{1}{4!}(\lambda t)^4 + \cdots$$

Similarly, $e^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \cdots$

where, A = System matrix (or) State Matrix

e^{At} = State transition Matrix

I = ~~Unit~~ Identity Matrix.

*** Computation of e^{At} using laplace transform method.**

Consider the state equation with nil input vector

$\dot{x}(t) = A\,x(t)$. ——— (1)

on taking the laplace transform for the eq. (1)

$\Rightarrow s\,X(s) - X(0) = A\,X(s)$.

$\Rightarrow s\,X(s) - A\,X(s) = X(0).$

$\Rightarrow (sI - A)\,X(s) = X(0).$

$\Rightarrow X(s) = [sI-A]^{-1} X(0).$

Now, apply inverse laplace transform to above equation:

$\Rightarrow x(t) = L^{-1} [sI-A]^{-1} X(0).$ —— ②

$\Rightarrow$ Compare equation ② with $x(t) = e^{At} x(0)$.

$$\boxed{e^{At} = L^{-1} [sI-A]^{-1}} \quad \text{—— ③}$$

$\Rightarrow L(e^{At}) = [sI-A]^{-1}$

$\Rightarrow L[\phi(t)] = [sI-A]^{-1}$

$$\Rightarrow \boxed{\phi(s) = [sI-A]^{-1}}$$

where, $\phi(s)$ is called the Resolvent Matrix.

problems.

1. Compute e^{At} by two Methods: $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Sol: given, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

Method-1:- $e^{At} = 1 + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \cdots\cdots$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$, $A^2 = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}.$

$A^3 = \begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}$, $A^4 = \begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}$.

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \frac{1}{4!}A^4t^4 + \cdots\cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}t + \frac{1}{2}t^2\begin{bmatrix} -2 & -3 \\ 6 & 7 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 6 & 7 \\ -14 & -15 \end{bmatrix}t^3 + \frac{1}{24}\begin{bmatrix} -14 & -15 \\ 30 & 31 \end{bmatrix}t^4 + \cdots$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ -2t & -3t \end{bmatrix} + \begin{bmatrix} \frac{-2}{2}t^2 & \frac{-3}{2}t^2 \\ \frac{6}{2}t^2 & \frac{7}{2}t^2 \end{bmatrix} + \begin{bmatrix} t^3 & \frac{7}{6}t^3 \\ \frac{-14}{6}t^3 & \frac{-15}{6}t^3 \end{bmatrix} + \begin{bmatrix} \frac{-14}{24}t^4 & \frac{-15}{24}t^4 \\ \frac{30}{24}t^4 & \frac{31}{24}t^4 \end{bmatrix} + \cdots\cdots$$

$$= \begin{bmatrix} 1-t^2+t^3-\frac{7}{12}t^4+\dots & t-\frac{3}{2}t^2+\frac{7}{6}t^3-\frac{5}{8}t^4+\dots \\ -2t+3t^2-\frac{7}{3}t^3+\frac{5}{4}t^4-\dots & 1-3t+\frac{7}{2}t^2-\frac{5}{2}t^3+\frac{31}{24}t^4+\dots \end{bmatrix}$$

Here, the each term in the above Matrix is the expansion of e^{At}.

The convergence of series is obtained by trail and error method.

$$e^{-t} = 1-t+\frac{1}{2!}t^2-\frac{1}{3!}t^3+\frac{1}{4!}t^4\dots = 1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\dots$$

$$e^{-2t} = 1-2t+\frac{1}{2!}(2t)^2-\frac{1}{3!}(2t)^3+\dots = 1-2t+2t^2-\frac{4}{3}t^3+\frac{2}{3}t^4-\dots$$

Now,

$$2e^{-t}-e^{-2t} = 2-2t+t^2-\frac{1}{3}t^3+\frac{1}{12}t^4-\dots-1+2t-2t^2+\frac{4}{3}t^3-\frac{2}{3}t^4+\dots$$

$$= 1-t^2+t^3-\frac{7}{12}t^4+\dots$$

$$e^{-t}-e^{-2t} = 1-t+\frac{1}{2}t^2-\frac{1}{6}t^3+\frac{1}{24}t^4-\dots-1+2t-2t^2+\frac{4}{3}t^3-\frac{2}{3}t^4+\dots$$

$$= t-\frac{3}{2}t^2+\frac{7}{6}t^3-\frac{5}{8}t^4+\dots$$

$$-2e^{-t}+2e^{-2t} = -2+2t-t^2+\frac{1}{3}t^3-\frac{1}{12}t^4+\dots+2-4t+4t^2-\frac{8}{3}t^3+\frac{4}{3}t^4-\dots$$

$$= -2t+3t^2-\frac{7}{3}t^3+\frac{5}{4}t^4-\dots$$

$$-e^{-t}+2e^{-2t} = -1+t-\frac{1}{2}t^2+\frac{1}{6}t^3-\frac{1}{24}t^4+\dots+2-4t+4t^2-\frac{8}{3}t^3+\frac{4}{3}t^4-\dots$$

$$= 1-3t+\frac{7}{2}t^2-\frac{5}{2}t^3+\frac{31}{24}t^4-\dots$$

$$\therefore e^{At} = \begin{bmatrix} 2e^{-t}-e^{-2t} & e^{-t}-e^{-2t} \\ -2e^{-t}+2e^{-2t} & -e^{-t}+2e^{-2t} \end{bmatrix}.$$

Method-2

given, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$e^{At} = \phi(t) = L^{-1}[SI-A]^{-1} \Rightarrow \phi(S) = [SI-A]^{-1}$.

Now, $[SI-A] = \begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$

$$|sI - A| = \begin{vmatrix} s & -1 \\ 2 & s+3 \end{vmatrix} = s(s+3)+2$$
$$= s^2+3s+2 = (s+2)(s+1).$$

Now, $$[sI-A]^{-1} = \frac{\text{adj}\,[sI-A]}{|sI-A|}$$ $$\left(\because \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \underset{\text{adj}}{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}} \right)$$

$$= \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix} \Big/ (s+1)(s+2)$$

$$\therefore \phi(t) = \phi(s) = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+2)(s+1)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

By doing the partial fractions we get,

$$\frac{s+3}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$
$$\Rightarrow s+3 = A(s+2) + B(s+1) \quad (\because s=-1)$$
$$\Rightarrow A = 2$$
for $s=-2$, $B = -1$.

$$\frac{1}{(s+1)(s+2)} = \frac{A}{(s+1)} + \frac{B}{(s+2)}$$
$$\Rightarrow 1 = A(s+2) + B(s+1)$$
for $s=-1$, $A = 1$
for $s=-2$, $B = -1$

$$\frac{-2}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$
$$-2 = A(s+2) + B(s+1)$$
for $s=-1$, $A = -2$
$s=-2$, $B = 2$

$$\frac{s}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$
$$s = A(s+2) + B(s+1).$$
for $s=-1$, $A = -1$
$s=-2$, $B = 2$

$$\therefore \phi(s) = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{(s+2)} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

On taking the inverse laplace transform we get,

$$\therefore e^{At} = \phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

(2) $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Compute e^{At}?

Sol. given, $A_1 = \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}$, $A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$

Here, $A = A_1 + A_2$

$$\therefore e^{At} = e^{(A_1+A_2)t} = e^{A_1t} \cdot e^{A_2t}$$

we know that, $e^{At} = \phi(t) = L^{-1}[sI - A]^{-1}$.

$$\therefore |sI - A_1| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} \sigma & 0 \\ 0 & \sigma \end{bmatrix}\right| = \begin{vmatrix} s-\sigma & 0 \\ 0 & s-\sigma \end{vmatrix}$$

$$= (s-\sigma)^2$$

Now, $$[sI - A]^{-1} = \begin{bmatrix} (s-\sigma) & 0 \\ 0 & (s-\sigma) \end{bmatrix} \Big/ (s-\sigma)^2$$

$$= \begin{bmatrix} \frac{1}{s-\sigma} & 0 \\ 0 & \frac{1}{s-\sigma} \end{bmatrix}.$$

$$\therefore e^{A_1t} = L^{-1}[sI - A_1]^{-1} = \begin{bmatrix} e^{+\sigma t} & 0 \\ 0 & e^{+\sigma t} \end{bmatrix}$$

$$e^{A_2t} = L^{-1}[sI - A_2]^{-1}$$

$$|sI - A_2| = \left|\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}\right| = \begin{vmatrix} s & -\omega \\ \omega & s \end{vmatrix}$$

$$= s^2 + \omega^2$$

$$\therefore [sI - A_2]^{-1} = \frac{\begin{bmatrix} s & \omega \\ -\omega & s \end{bmatrix}}{s^2+\omega^2}$$

$$= \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

Now, $e^{A_2 t} = L^{-1}[sI - A_2]^{-1}$

$$= L^{-1} \begin{bmatrix} \frac{s}{s^2+\omega^2} & \frac{\omega}{s^2+\omega^2} \\ \frac{-\omega}{s^2+\omega^2} & \frac{s}{s^2+\omega^2} \end{bmatrix}$$

$$= \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$e^{At} = e^{A_1 t} \cdot e^{A_2 t}$$

$$= \begin{bmatrix} e^{+\sigma t} & 0 \\ 0 & e^{+\sigma t} \end{bmatrix} \cdot \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}$$

$$= \begin{bmatrix} e^{+\sigma t}\cos\omega t & e^{+\sigma t}\sin\omega t \\ -e^{+\sigma t}\sin\omega t & e^{+\sigma t}\cos\omega t \end{bmatrix}.$$

3. A linear time invariant system is characterized by homogenius state equation.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Compute the solution of homogenius equation.

Sol given, $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution of homogenius equation $= x(t) = e^{At} x_0$.

where, $e^{At} = L^{-1}[sI - A]^{-1}$

$$\Rightarrow |sI - A| = \left| \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \right| = \left| \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix} \right|$$

$$= (s-1)^2$$

$$[sI-A]^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{+1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$\left\{ \because \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \right\}$$

$$e^{At} = \phi(t) = L^{-1}[sI-A]^{-1}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{+1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} & 0 \\ te^{+t} & e^{t} \end{bmatrix}$$

$$\therefore e^{At} = \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix}$$

Now, Solution $= x(t) = e^{At}x_0$

$$= \begin{bmatrix} e^{t} & 0 \\ te^{t} & e^{t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} \\ te^{t} \end{bmatrix}.$$

* Concept of Controllability and observability :-

Controllability :-

The Controllability verifies the usefullness of a state variable. In the Controllability Test, we can find, whether the state variable can be Controlled to achieve the desired output & not. After determining the state Model, the Controllability of a state variable is verified.

* Definition of Controllability :-

A System is said to be Completely Controllable if it is possible to transfer the system state from any initial state $x_0(t)$ to any other desired state $x(t_d)$ in specified finite time.

The Controllability of a state Model can be tested by Kalman's Method and Gilbert's Method.

i, Gilbert's Method :-

⇒ when the System Matrix has distinct eigen values, normal diagonalization is possible. In this the System Matrix can be diagnolised and the state Model is Converted to Canonical form. → [when Repeated Eigen values obtained]

Consider the state Model as below,

$$\dot{x} = AX + BU$$

$$y = CX + DU.$$

The state model can be converted to canonical form by a transformation $X = MZ$.

where, M = Modal matrix, Z = State variable vector.

$\therefore \dot{X} = AX + BU$

$\Rightarrow M\dot{Z} = AMZ + BU$

$\Rightarrow \dot{Z} = M^{-1}AMZ + M^{-1}BU$

$\therefore \boxed{\dot{Z} = \Lambda Z + \bar{B}U}$ $\left[\because \Lambda = M^{-1}AM, \quad \bar{B} = M^{-1}B\right]$

Similarly $Y = CX$ [$\because$ generally $D = 0$]

$Y = CMZ$

$\therefore \boxed{Y = \bar{C}Z}$ $[\because \bar{C} = CM]$

Now, the Necessary and Sufficient Condition for Complete Controllability is that, the Matrix $\bar{B}$ must have no rows with all zeros.

$\Rightarrow$ When the System Matrix has Repeated Eigen Values.

In this case, the System Matrix cannot be diagonalized but can converted to Jordan Canonical form.

Consider State Model $\dot{X} = AX + BU$

$Y = CX + DU$

The State Model can be transformed to Jordan Canonical form by a transformation $X = MZ$

∴ The transformed Model is given as Similar to above Case

$$\dot{Z} = JZ + \bar{B}U$$

$$Y = \bar{C}Z + DU$$

where, $J = M^{-1}AM$

$\bar{B} = M^{-1}B$

$\bar{C} = CM.$

* Kalman's Method for Testing Controllability :-

Consider a state Equation $\dot{x} = Ax + BU$, For this System, a Composite matrix Q_c Can be formed Such that,

$$\boxed{Q_c = [B \quad AB \quad A^2B \; .... \; A^{n-1}B]}$$

where, n = order of the System (or) no. state variables.

In this Case, the System is Completely Controllable, if the rank of the Composite matrix is "n".

The Rank of matrix Q_c is n, if the determinant of Composite Matrix Q_c is non zero, i-e, $|Q_c| \neq 0$.

* Observability :-

Definition :-

A System is said to be Completely observable if every state $x(t)$ can be Completely identified by measurements of output $y(t)$ over a finite time interval.

i, Gilbert's Method :-

Consider the state Model, $\dot{x} = AX + BU$

$$Y = CX + DU.$$

The state Model can be transformed to Canonical (or) Jordan Canonical form by a transformation $X = MZ$.

Now the transformed state Model is,

$$\dot{z} = \Lambda Z + \bar{B}U \quad \text{(or)} \quad \dot{z} = JZ + \bar{B}U$$

$$y = \bar{C}Z + DU \qquad\qquad y = \bar{C}Z + DU$$

where, $\Lambda = M^{-1}AM$, if Eigen values are distinct

$J = M^{-1}AM$, if Eigen values are Repeated

$\bar{B} = M^{-1}B$ and $\bar{C} = CM$.

Now, the necessary and sufficient Condition for Complete observability is that none of the Columns of the Matrix $\bar{C}$ is zero.

ii) **Kalman's Test for Observability :-**

Consider a State Model $\dot{x} = AX + BU$
$Y = CX + DU.$

For this State Model, a Composite Matrix Q_o Can be formed Such that,

$$\boxed{Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \ldots\ldots (A^T)^{n-1} C^T\right]}$$

where, n = order of System (or) no. of State Variables.

In this Case, the System is Completely observable if the determinant of Q_o is non- zero i.e, $|Q_o| \neq 0$.

* **Problems :-**

① . A linear time invariant System is described by State Model,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} u \quad \& \quad Y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Determine whether Completely Controllable and observable & not.

Sol. i, **Gilberts Method :-**

$$|\lambda I - A| = \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \right|$$

$$= \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ 6 & 11 & \lambda+6 \end{vmatrix}$$

$$= \lambda\left[\lambda(\lambda+6)+11\right] + 1\times 6 = 0$$

$$= \lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0$$

$\therefore \lambda = -1, -2, -3.$ (distinct eigen values)

$$[\lambda_1 I - A] = \begin{bmatrix} -1 & -1 & 0 \\ 0 & -1 & -1 \\ 6 & 11 & 5 \end{bmatrix}$$

Now, $C_{11} = 6$, $C_{12} = -6$, $C_{13} = 6$ $\therefore x_1 = \begin{bmatrix} 6 \\ -6 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Similarly,

$$[\lambda_2 I - A] = \begin{bmatrix} -2 & -1 & 0 \\ 0 & -2 & -1 \\ 6 & 11 & 4 \end{bmatrix}$$

Now, $C_{11} = 3$, $C_{12} = -6$, $C_{13} = 12$ $\therefore x_2 = \begin{bmatrix} 3 \\ -6 \\ 12 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}$

Now, $$[\lambda_3 I - A] = \begin{bmatrix} -3 & -1 & 0 \\ 0 & -3 & -1 \\ 6 & 11 & 3 \end{bmatrix}$$

$\therefore C_{11} = 2$, $C_{12} = -6$, $C_{13} = 18$, $\therefore x_3 = \begin{bmatrix} 2 \\ -6 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$

$\therefore$ Modal Matrix $M = [x_1 \ x_2 \ x_3]$

$$M = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$\therefore M^{-1} = \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ +1 & 1.5 & 0.5 \end{bmatrix}.$$

4

$$\Lambda = M^{-1}AM$$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}.$$

$$\bar{B} = M^{-1}B$$

$$= \begin{bmatrix} 3 & 2.5 & 0.5 \\ -3 & -4 & -1 \\ 1 & 1.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

$$\bar{C} = CM$$

$$= \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & -2 & -3 \\ 1 & 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

Now, It is observed that all the Elements of $\bar{B}$ Matrix are not all zeros. Hence the System is Completely Controllable

Similarly.

It is observed that the Elements of Columns of $\bar{C}$ Matrix are not all zeros. ∴ System is Completely Observable

Alternate Method :-

Kalmans test for Controllability :-

$$Q_c = [B \quad AB \quad A^2B]$$

given, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix}$$

Now, $AB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -12 \end{bmatrix}$

$$A^2B = \begin{bmatrix} 0 & 0 & 1 \\ -6 & -11 & -6 \\ 36 & 60 & 25 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -12 \\ 50 \end{bmatrix} = \begin{bmatrix} 1 \\ -6 \\ 25 \end{bmatrix}$$

$$\therefore Q_c = \left| \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & -6 \\ 2 & -12 & 25 \end{bmatrix} \right|$$

$$= 1(0-4) = -4 \neq 0.$$

$\therefore$ System is Completely Controllable

Similarly, Kalman's test for observability :-

$$Q_o = \left[C^T \quad A^T C^T \quad (A^T)^2 C^T \right]$$

given, $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}$; $A^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$

$$(A^T)^2 = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix}$$

$\because$ given $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$

$C^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$A^T C^T = \begin{bmatrix} 0 & 0 & -6 \\ 1 & 0 & -11 \\ 0 & 1 & -6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & -6 & 36 \\ 0 & -11 & 60 \\ 1 & -6 & 25 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\therefore Q_o = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= 1(1-0) - 0 + 0$$

$$= 1 \neq 0.$$

$\therefore |Q_o| \neq 0$ then the system is said to be completely observable.

② Convert the given Matrix into Canonical form.

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix}$$

Sol. To find eigen values,

$$|[\lambda I - A]| = 0 \Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \right| = 0$$

$$= \begin{vmatrix} \lambda-4 & -1 & 2 \\ -1 & \lambda & -2 \\ -1 & 1 & \lambda-3 \end{vmatrix}$$

$$= \lambda^3 - 7\lambda^2 + 15\lambda - 9 = 0$$

$\lambda = 1, 3, 3.$ (Repeated values) $\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 3.$

$$[\lambda_1 I - A] = \begin{bmatrix} -3 & -1 & 2 \\ -1 & 1 & -2 \\ -1 & 1 & -2 \end{bmatrix}$$

$C_{11} = 0,\ C_{12} = 0,\ C_{13} = 0$; $\therefore X_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

It gives null solution.

let, C_{21}, C_{22}, C_{23} be co-factors of $[\lambda_1 I - A]$ along II^{nd} row.

$C_{21} = -(0)$ $= 0$	$C_{22} = +(8)$ $= 8$	$C_{23} = -(-4)$ $= 4$

$$\therefore X_1 = \begin{bmatrix} 0 \\ 8 \\ 4 \end{bmatrix} =$$

$$[\lambda_2 I - A] = \begin{bmatrix} \lambda_2 - 4 & -1 & 2 \\ -1 & \lambda_2 & -2 \\ -1 & 1 & \lambda_2 - 3 \end{bmatrix}$$

$$C_{11} = +\begin{vmatrix} \lambda_2 & -2 \\ 1 & \lambda_2 - 3 \end{vmatrix} = \lambda_2(\lambda_2 - 3) + 2 = \lambda_2^2 - 3\lambda_2 + 2$$

$$C_{12} = -\begin{vmatrix} -1 & -2 \\ -1 & \lambda_2 - 3 \end{vmatrix} = (-1)\left[-1(\lambda_2 - 3) - 2\right] = \lambda_2 - 1$$

$$C_{13} = +\begin{vmatrix} -1 & \lambda_2 \\ -1 & 1 \end{vmatrix} = -1 + \lambda_2 = \lambda_2 - 1.$$

$$\therefore X_2 = \begin{bmatrix} \lambda_2^2 - 3\lambda_2 + 2 \\ \lambda_2 - 1 \\ \lambda_2 - 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \qquad [\because \lambda_2 = 3]$$

Similarly, eigen vector X_3 is given by,

$$X_3 = \begin{bmatrix} \frac{d}{d\lambda_2} C_{11} \\ \frac{d}{d\lambda_2} C_{12} \\ \frac{d}{d\lambda_2} C_{13} \end{bmatrix} = \begin{bmatrix} \frac{d}{d\lambda_2}(\lambda_2^2 - 3\lambda_2 + 2) \\ \frac{d}{d\lambda_2}(\lambda_2 - 1) \\ \frac{d}{d\lambda_2}(\lambda_2 - 1) \end{bmatrix} = \begin{bmatrix} 2\lambda_2 - 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$\therefore M = [X_1 \quad X_2 \quad X_3]$$

$$= \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \Rightarrow M^{-1} = \begin{bmatrix} 0 & 0.25 & -0.25 \\ -0.25 & -0.75 & 1.5 \\ 0.5 & 0.5 & -1 \end{bmatrix}$$

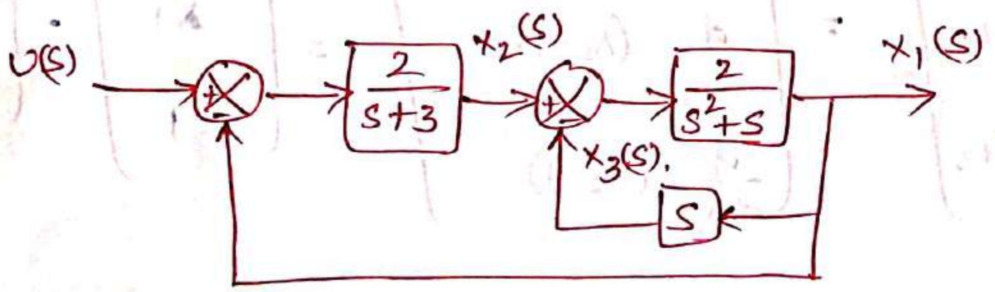
$$\therefore \cancel{P} = \cancel{\begin{bmatrix} 0 & 1 & 1 \\ -0.5 & -1.5 & 3 \\ 0.5 & 0.5 & -1 \end{bmatrix}} \quad \cancel{\begin{bmatrix} 0 & 0.142 & -0.142 \\ -0.25 & -0.107 & 0.857 \\ 0.5 & 0.071 & -0.571 \end{bmatrix}}$$

$$\therefore M^{-1}AM = \underbrace{\begin{bmatrix} \cancel{0} & \cancel{1} & \cancel{1} \\ \cancel{-0.5} & \cancel{1.5} & \cancel{3} \\ \cancel{0.5} & \cancel{0.5} & \cancel{-1} \end{bmatrix}}_{M^{-1}} \begin{bmatrix} 4 & 1 & -2 \\ 1 & 0 & 2 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 2 & 3 \\ 8 & 2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

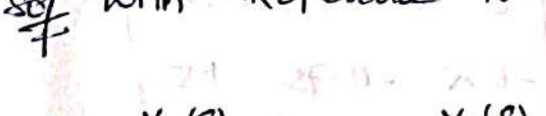
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \text{Jordan Block.}$$

$$\therefore J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$ Canonical form of given Matrix "A".

(3) For the given system, write the equations for $\dot{x}_1, \dot{x}_2$ & $\dot{x}_3$ Determine whether the system is Controllable and Observable (or) not.



Sol with Reference to the above figure



$$\Rightarrow X_1(S) = [X_2(S) - X_3(S)]\left[\frac{2}{s^2+s}\right]$$

$$\Rightarrow (s^2+s)\, X_1(S) = 2X_2(S) - 2X_3(S)$$

$$\Rightarrow s^2 X_1(S) + s X_1(S) = 2X_2(S) - 2X_3(S)$$

taking I.L.T $\Rightarrow \ddot{x}_1 + \dot{x}_1 = 2x_2 - 2x_3.$ —— (1)

Similarly, $X_3(s) \leftarrow \boxed{s} \leftarrow X_1(s)$

$$\Rightarrow X_3(s) = s\,X_1(s)$$

taking I.L.T $\Rightarrow x_3 = \dot{x}_1$ —— (2)

$$U(s) - X_1(s) \longrightarrow \boxed{\frac{2}{s+3}} \longrightarrow X_2(s).$$

$$\Rightarrow X_2(s) = \left[U(s) - X_1(s)\right]\left(\frac{2}{s+3}\right)$$

$$\Rightarrow X_2(s)(s+3) = 2U(s) - 2X_1(s)$$

$$\Rightarrow sX_2(s) + 3X_2(s) = 2U(s) - 2X_1(s)$$

taking I.L.T. $\Rightarrow \dot{x}_2 + 3x_2 = 2u - 2x_1$ —— (3)

(1) $\Rightarrow \ddot{x}_1 + \dot{x}_1 = 2x_2 - 2x_3$

(2) $\Rightarrow \dot{x}_1 = x_3$

(3) $\Rightarrow \dot{x}_2 = -2x_1 - 3x_2 + 2u$

As we got $\dot{x}_1 = x_3$

$$\Rightarrow \ddot{x}_1 = \dot{x}_3.$$

$\therefore$ (1) $\Rightarrow \dot{x}_3 + x_3 = 2x_2 - 2x_3$

$$\Rightarrow \dot{x}_3 = 2x_2 - 3x_3.$$

$\therefore$ state model,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & +2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} u; \quad y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

from the state model $A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$.

Characteristic Equation $|\lambda I - A| = 0$

$$\Rightarrow \left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} \lambda & 0 & -1 \\ 2 & \lambda+3 & 0 \\ 0 & -2 & \lambda+3 \end{vmatrix} = 0 \quad \Rightarrow \lambda^3 + 6\lambda^2 + 9\lambda + 4 = 0.$$

$\therefore \lambda = -1, -1, -4.$ (Repeated Values)

$\lambda_3 = -4, \quad \lambda_2 = -1, \quad \lambda_1 = -1.$

$$[\lambda_1 I - A] = \begin{bmatrix} \lambda_1 & 0 & -1 \\ 2 & \lambda_1+3 & 0 \\ 0 & -2 & \lambda_1+3 \end{bmatrix}$$

$$c_{11} = + \begin{vmatrix} \lambda_1+3 & 0 \\ -2 & \lambda_1+3 \end{vmatrix} = \lambda_1^2 + 6\lambda_1 + 9$$

$$c_{12} = - \begin{vmatrix} 2 & 0 \\ 0 & \lambda_1+3 \end{vmatrix} = -2\lambda_1 - 6.$$

$$c_{13} = + \begin{vmatrix} 2 & \lambda_1+3 \\ 0 & -2 \end{vmatrix} = -4.$$

$$\therefore x_1 = \begin{bmatrix} \lambda_1^2 + 6\lambda_1 + 9 \\ -2\lambda_1 - 6 \\ -4 \end{bmatrix} = \begin{bmatrix} 4 \\ -4 \\ -4 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \frac{d}{d\lambda_1} c_{11} \\ \frac{d}{d\lambda_1} c_{12} \\ \frac{d}{d\lambda_1} c_{13} \end{bmatrix} = \begin{bmatrix} 2\lambda_1 + 6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix}.$$

$[\lambda_3 I - A] = \begin{bmatrix} -4 & 0 & -1 \\ 2 & -1 & 0 \\ 0 & -2 & -1 \end{bmatrix}$ (8)

$c_{11} = 1, \quad c_{12} = -(-2) = 2 \quad c_{13} = -4$

$X_3 = \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix}.$

$\therefore$ Modal Matrix $M = \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix}$

$M^{-1} = \begin{bmatrix} -0.111 & -0.222 & -0.138 \\ 0.333 & 0.166 & 0.166 \\ 0.111 & 0.222 & -0.111 \end{bmatrix}.$

Jordan Block.

$J = M^{-1} A M$

$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ [on Symplifying, we get,],

$\bar{B} = M^{-1} B$

$= \begin{bmatrix} -0.111 & -0.222 & -0.138 \\ 0.333 & 0.166 & 0.166 \\ 0.111 & 0.222 & -0.111 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} -0.444 \\ 0.232 \\ 0.444 \end{bmatrix}$

$\bar{C} = CM = [1 \ 0 \ 0] \begin{bmatrix} 4 & 4 & 1 \\ -4 & -2 & 2 \\ -4 & 0 & -4 \end{bmatrix} = [4 \ 4 \ 1].$

From the $\bar{B}$ Matrix no Element is zero, i.e, all the Elements of $\bar{B}$ is non zeros. Hence System is Completely Controllable

Similarly, from $\bar{C}$ Matrix, all the Elements are non-zeros.

$\therefore$ System is Completely observable.

Alternate Method :-

for Controllability, Kalman's method

$$Q_c = [B \quad AB \quad A^2B]$$

$$AB = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & +2 & -3 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 4 \end{bmatrix}$$

$$A^2B = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$$

$$\left\{ \because A^2 = A \cdot A = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 2 & -3 \\ 6 & 9 & -2 \\ -4 & -12 & 9 \end{bmatrix} \right\}$$

$$= \begin{bmatrix} 4 \\ 18 \\ -24 \end{bmatrix}$$

$$\therefore Q_c = \left| \begin{bmatrix} 0 & 0 & 4 \\ 2 & -6 & 18 \\ 0 & 4 & -24 \end{bmatrix} \right|$$

$$= 4\,[(8) - 0] = 32$$

$\therefore |Q_c| \neq 0$.

$\therefore$ given System is Completely Controllable for Observability, Kalmans Method.

$$Q_o = \begin{bmatrix} C^T & A^T C^T & (A^T)^2 C^T \end{bmatrix}$$

$$C^T = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

$$(A^T) = \begin{bmatrix} 0 & 0 & 1 \\ -2 & -3 & 0 \\ 0 & 2 & -3 \end{bmatrix}^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$(A^T)^2 = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix}$$

$$A^T C^T = \begin{bmatrix} 0 & -2 & 0 \\ 0 & -3 & 2 \\ 1 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(A^T)^2 C^T = \begin{bmatrix} 0 & 6 & -4 \\ 2 & 9 & -12 \\ -3 & -2 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \end{bmatrix}$$

$$\therefore Q_o = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{bmatrix}$$

$$\Rightarrow |Q_o| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 1 & -3 \end{vmatrix}$$

$$= 1(-2) = -2 \neq 0.$$

$\therefore |Q_0| \neq 0$.

$\therefore$ given System is Completely Observable